Economic Modelling 126 (2023) 106376

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.journals.elsevier.com/economic-modelling

Information and optimal trading strategies with dark pools ☆

Anna Bayona [a], Ariadna Dumitrescu [a], Carolina Manzano [b,*]

[a] ESADE Business School, Univ. Ramon Llull, Av. Pedralbes, 60-62, Barcelona, 08034, Spain
[b] Universitat Rovira i Virgili and CREIP, Av. de la Universitat 1, Reus, 43204, Spain

ARTICLE INFO

JEL classification:
G12
G14
G18

Keywords:
Dark liquidity
Limit order book
Adverse selection
Market performance

ABSTRACT

This paper examines the effects of the competition between asset trading venues with different levels of transparency: an opaque dark pool alongside a transparent exchange organized as a limit order book (two-venue market). In a model with asymmetric information, we compare traders' strategies and market performance in the two-venue market with that of a single-venue market (trading only in the exchange). We show that price informativeness is lower in the two-venue market when informed traders migrate to the dark pool and uninformed investors remain in the exchange. We also find that when orders migrate to the dark pool in the first period, market liquidity is lower (higher) in the two-venue market for high (low) fundamental volatility stocks as traders migrating to the dark pool would have demanded (supplied) liquidity in the exchange. Finally, the expected profits of informed traders are never lower in the two-venue market, but this may not always be true for uninformed traders.

1. Introduction

In today's financial markets traders have access to competing trading venues for buying or selling assets with different levels of transparency. In addition to transparent exchanges, market participants can also trade in opaque trading venues such as dark pools. In December 2022, dark pools accounted for 13.75% of the US equity volume in the United States, and 7.50% of the total value traded in European markets.[1] Dark pools often foster price improvement in relation to exchanges, but pose execution risks. In this context, information asymmetries play a fundamental role in investors' decision of where to trade and in the price discovery process. Therefore, a better understanding of the competition between an exchange and a dark pool with the presence of asymmetric information is essential.

The extant empirical literature finds mixed results regarding the impact of dark trading on market quality. Furthermore, the theoretical literature has not fully addressed how prices evolve when a limit order book trades alongside a dark pool, especially in the presence of long-lived and asymmetric information. In this paper, we aim at filling this gap by focusing on the following research questions: What are the optimal trading strategies for the different types of investors when a transparent exchange coexists with an opaque dark pool? How is the order placement decision influenced by the fact that rational traders learn from the state of the limit order book and optimize their behavior? How does the co-existence of the dark pool and the exchange affect market quality and investors' expected profits?

To answer these questions we propose a two-venue market model where a dark pool and an exchange co-exist, and compare traders'

☆ We thank the Editor, Associate Editor and Referees for providing valuable suggestions to the manuscript. The paper has also benefited from comments of Vladimir Asriyan, Michael Brolley, Jordi Caballé, Julio Crego, Martin Haferkorn, Javier Gil-Bazo, Chinmay Jain, Tomy Lee, Laurence Lescourret, Robin Mess, Sophie Moinas, Roberto Pascual, José Penalva, Eghbal Rahimikia, Andreea Vaduva, Xavier Vives, and seminar participants at the Finance Theory Group Summer School 2017, ERMAS 2017, CEPR European Summer Symposium in Financial Markets 2017, Finance Forum 2017, CEPR-Imperial-Plato Conference, Swiss Society for Financial Market Research, 42nd Symposium of Economic Analysis, 2nd Catalan Economic Society Conference, 2019 Conference of the Society for the Advancement of Economic Theory, EARIE 2019, 3rd SAFE Market Microstructure Conference, Financial Management Association 2019 Annual Meeting, 10th Annual Financial Market Liquidity Conference, 37th International Conference of the French Finance Association, MIFID II three years later - IE Business School/CNMV Conference, Consob-ESMA-Bocconi Seminar Series, Toulouse School of Economics, ESSEC Business School, Universitat de les Illes Baleares, and ESADE Business School. The authors acknowledge financial support from the Government of Spain (AEI/FEDER, UE PID2019-105982GBI00, PID2021-123748NB-I00, PR2015-00645), the Government of Catalonia (2017-SGR-770, 2017-SGR-640), Universitat Rovira i Virgili (2019PFR-URV-B2-53), Banc Sabadell, and the URL-La Caixa Foundation. Any errors are our own responsibility.

* Corresponding author.
E-mail addresses: anna.bayona@esade.edu (A. Bayona), ariadna.dumitrescu@esade.edu (A. Dumitrescu), carolina.manzano@urv.cat (C. Manzano).

[1] For the US, see "Let there be light, "published by Rosenblatt Securities in January 2023, while for European markets, see "Liquidity XYT View," December 2022 from *big xyt*.

https://doi.org/10.1016/j.econmod.2023.106376
Received 6 September 2022; Received in revised form 15 May 2023; Accepted 17 May 2023
Available online 22 May 2023

optimal strategies and market quality indicators in this two-venue market with those in a model in which traders only have access to the exchange (single-venue market). In our model, there is one risky asset and the information about it is asymmetric and long-lived. Traders arrive randomly and sequentially to the market and can be of two types: rational traders, who strategically choose whether or not to trade, and if they trade, they simultaneously select the venue and the type of order that maximize profits given their information; and liquidity traders, who participate in the market for liquidity reasons, and submit market orders only to the exchange to ensure immediate execution. Rational traders can submit several order types: a market order or a limit order to the exchange, or a dark pool order. In addition, rational traders may be informed if they know the liquidation value of the asset perfectly, or (privately) uninformed if they know only the distribution of the liquidation value of the asset conditional on public information. In the second period, an uninformed trader may learn about the liquidation value of the asset from the changes in the limit order book.

When submitting an order to the dark pool a trader faces a trade-off between price improvement and immediacy. If the order is not immediately executed, then it is either cancelled or re-routed to the exchange in the following period. Therefore, there is execution risk, meaning that the order cannot be executed in the dark pool and that, if the order returns to the exchange, the price has moved unfavorably. Although many types of dark pools exist (see Section 2 for a review), we focus on a dark pool that executes orders at the midpoint of the best bid and ask prices quoted on the exchange's limit order book. In short, this model allows us to study the optimal decisions of different types of traders when an exchange coexists with a dark pool that offers maximal price improvement, and see how the decision evolves when traders can learn from the exchange's limit order book.

We first analyze traders' equilibrium strategies in the single-venue market (our benchmark) and show that, in the first period, traders demand or supply liquidity in the exchange or do not trade depending on market conditions. Specifically, informed traders demand (supply) liquidity for high (low) fundamental volatility stocks, while uninformed traders supply liquidity or do not to trade depending on the degree of adverse selection they face. In the two-venue market model, we show that in the first trading period, an informed trader finds the dark pool more attractive than the exchange when the dark pool execution probability and price improvement are sufficiently high. In contrast, an uninformed trader does not go to the dark pool in the first trading period since the price improvement is not sufficient to induce a trade in the opaque venue. Nevertheless, the existence of a dark pool alongside the exchange may change the uninformed trader's optimal submission strategy from not trading to supplying liquidity in the exchange when adverse selection in the exchange decreases. In such a case, there is order flow segmentation in the first trading period. In the second period, we find that both informed and uninformed traders migrate to the dark pool if the probability of execution in the dark pool is high enough and the price improvement is significant.

We then analyze how the migration of orders to the dark pool impacts market performance. We show that dark trading has a negative impact on price informativeness for all stocks, in both periods, except when in the second period there is fragmentation of the order flow and informed traders choose to trade in the exchange, while uninformed traders choose to trade in the dark pool. We also show that market liquidity increases (decreases) initially for high (low) fundamental volatility stocks as traders migrating to the dark pool would have demanded (supplied) liquidity in the exchange. Moreover, in the two-venue market, expected profits of informed traders are always higher or equal than in the single-venue market. For uninformed traders this result holds in the first period and also in the second period when there is high fundamental volatility and low adverse selection. Therefore, we find that the effects of dark trading may differ between trading periods. This is in contrast to other theoretical frameworks, and is due to the fact that rational traders learn from the state of the limit order book and optimally change their trading behavior.

Our work contributes to the growing body of theoretical research on the effects of competition between exchanges and dark pools.[2] In a static set-up, Hendershott and Mendelson (2000) find that a crossing network with midpoint pricing that competes with a dealer market has both a positive liquidity externality and a negative crowding externality, leading to ambiguous effects on market quality that depend on the insider's informational advantage. Degryse et al. (2009) show that the same positive and negative externalities remain in a dynamic setup and analyze how welfare and the order flow dynamics depend on the degree of market transparency. Menkveld et al. (2017) propose the pecking order hypotheses of trading venues, which conjectures that investors place midpoint dark pools at the top (since they are low-cost and low-immediacy) and lit markets at the bottom (since they are high-cost and high-immediacy), and find empirical support for it. Ye and Zhu (2020) study how an informed trader splits the order between a dark pool and a dealer market, and show that trades are more aggressively in the dark pool than on the exchange.[3]

Our paper is more closely related to Zhu (2014), Buti et al. (2017) and Brolley (2020). Like in Zhu (2014), we examine the role of asymmetric information in competing trading venues. However, we model the competition of a dark pool with a limit order book instead of a dealer market; therefore, in our framework, traders can both demand liquidity and supply liquidity to the exchange. Moreover, in contrast to Zhu (2014), we propose a two-period model, which allows for the first time to examine how information is gradually incorporated in the limit order book, and how traders' strategies reflect this change. Interestingly, under a particular parameter configuration, we find the same result as Zhu (2014) that dark pools improve price informativeness (when in the second trading period the informed stays in the exchange and the uninformed trades in the dark pool). In contrast, we show that when market conditions are such that the informed trader migrates to the dark pool and the uninformed stays in the exchange, the existence of the dark pool harms price informativeness.[4]

Buti et al. (2017) and Brolley (2020) examine the competition between a fully transparent limit order book and a dark pool. In a symmetric information setup with private values, Buti et al. (2017) show that the introduction of a dark pool that competes with an illiquid limit order book is, on average, associated with trade creation, wider spreads, lower depth, and welfare deterioration. To complement their work, we introduce asymmetric information in a common value setup and find the same results as in Buti et al. (2017) for low fundamental volatility stocks (when the information held by the informed trader is of low value) in the first trading period. However, since traders learn from prices in the second trading period, our market quality results differ fundamentally. In a model with asymmetric information, Brolley (2020) shows that the impact of dark trading on market quality depends on the relative price improvement of dark orders over limit orders. In contrast to Brolley (2020), we develop a model in which the dark pool reference price is the midpoint of the exchange and where, in general,

[2] Glosten (1994), Chakravarty and Holden (1995), Seppi (1997), Kaniel and Liu (2006) emphasize the role of asymmetric information in the order submission strategies' choice in a single trading venue. Parlour (1998), Foucault (1999), Parlour and Seppi (2003), Foucault et al. (2005), Goettler et al. (2009), Rosu (2009), Brolley and Malinova (2021), and Riccó et al. (2020) study the optimal choice of order type in dynamic models.

[3] Our research is also related to two other broader strands of the literature: competition between multiple trading venues (see Gomber et al., 2016 for a review of the literature) and transparency (Biais, 1993; Madhavan, 1995; Frutos and Manzano, 2002, 2005; Dumitrescu, 2010; Boulatov and George, 2013; Hendershott et al., 2022, among others).

[4] Ye (2011) finds that adding a dark pool alongside a dealer market always reduces price informativeness if the uninformed is restricted to trading in the exchange.

the market conditions in the two trading periods differ. We characterize how the effects of a dark pool that competes with an exchange on market quality depend on the market quality indicator, trading period, and stock and trader characteristics. The differences in both trading periods emerge because in our two-period trading model with long-lived information, an uninformed trader uses the prices in the limit order book to extract information about the common liquidation value of the asset.

2. Overview of dark pools around the world

The landscape for equity trading venues in the US, Europe, Canada and Australia is nowadays fragmented. Trading on-the-book indicates that participants trade in exchanges or other regulated platforms (as opposed to off-the-book which involves trading away from exchanges or regulated platforms in a bilateral way such as over-the-counter). On-the-book intra-day trading involves lit trading venues (mainly exchanges) that offer pre-trade transparency, that is, they provide information about the limit order book, and dark trading venues that do not publish any information about buy and sell orders looking for execution.

There are several other dimensions which characterize dark pools (ownership, pricing, accessibility, intermediation, order types, etc.). Zhu (2014) discusses three main types of dark pools: (i) agency broker and exchanged-owned dark pools, which act as of their customers and derive prices from the exchange; (ii) broker–dealer dark pools that trade for their clients (the order flow may also include proprietary trading) and offer a variety of order types, such as market orders, limit orders or pegged orders (for example, pegged at the midpoint of exchange prices) which may involve some price discovery; and (iii) independent electronic market makers that trade on their own account. Another relevant dimension of dark pool orders is the time-in-force (instruction that specifies how long an order will remain active before it is executed or expires). Common types of time-in-force instructions are day orders which are cancelled at the end of the trading day; or immediate-or-cancel (IOC) orders that require an immediate execution, otherwise the order is cancelled or is routed to the exchange (Petrescu and Wedow, 2017; Bank of America Securities, 2021). In our model, investors' orders submitted to the dark pool are all IOCs. Dark pools typically offer price improvements over displayed quotes on public exchanges. As a result, a main feature of dark pools is the pricing mechanism that they use and the % of price improvement. Our model uses midpoint pricing and so it applies to dark pools that use this.

The regulation of dark pools has evolved over time and varies across different geographic areas. Here we describe the main types of dark pools in various geographies, underlying two dimensions: midpoint pricing and ownership/agency (which gives information about the liquidity providers). In the US, Regulation National Market System (reg NMS) and Regulation Alternative Trading Systems (reg ATS) constitute the regulatory framework, which fosters competition among trading venues. The trading venues landscape is, therefore, very fragmented in the US, with broker–dealer dark pools dominating the landscape (such as UBS, Goldman Sachs, Morgan Stanley, JP Morgan, and Credit Suisse), followed by agency broker or exchange dark pools (such as Intelligent Cross and Level), followed by independent market makers such as Virtu.[5] Regulations do not restrict trading at the midpoint of the national best bid and offer (NBBO) quote of all trading venues, and dark pool orders can be executed at the NBBO, at the midpoint or inside NBBO excluding midpoint. However, some dark pool business models (typically called crossing networks) only execute orders at the midpoint. Examples of these include Intelligent Cross, Instinet, and Liquidnet, among others. Brolley (2020) argues that 26% of the dark pools use a midpoint crossing mechanism using data from 2016–2017. Moreover, many broker–dealer dark pools allow traders to use the midpoint price peg, with a significant trading volume executed at the midpoint.[6]

In Europe, the Markets in Financial Instruments Directive (MiFID II), which was implemented in 2018, sets the major regulatory framework for trading venues, which includes Multilateral Trading Facilities (MTF). Importantly, the MiFID II rules impose that, under the Reference Price Waiver all orders are executed at the midpoint of the bid and the ask of the lit venue. According to *big xyt*, a data analytic solutions company, the main trading venues market shares (as a % of the total value traded in Euros) in December 2022 were as follows: 55% for lit limit order books, 7.5% for dark pools, and the rest was split between auctions and off-the-book trading. In terms of ownership in the EMEA (Europe, Middle-East and Africa), examples of agency broker or exchanged-owned dark pools are CBOE (with the highest dark market share), Turquoise, SIX Swiss Exchange or Liquidnet; while examples of broker–dealer dark pools include UBS MTF and Sigma X MTF (owned by Goldman Sachs); and an example of an independent market-maker is Posit, which is owned by Virtu.[7]

Both in Canada and in Australia, the current regulatory framework involves minimum price improvement rules, effectively resulting in most of the trading occurring at the midpoint of the exchange (see Comerton-Forde and Putniņš, 2015; Comerton-Forde et al., 2018). In both of these markets, the predominant dark pools are exchange-owned MatchNow (currently owned by CBOE) and Nasdaq Dark in Canada, while in Australia, the market is dominated by the Australian Securities Exchange (ASX) and CBOE Australia.[8] There are also broker–dealer dark pools, but these represent a small market share.

3. Model

This section presents the model of intra-day trading, where market participants choose the venue and order type. A summary of the notation used is provided in Appendix A.

The asset. We consider a market in which a single risky asset is traded. The liquidation value of the asset, $\widetilde{V}$, may take two values, $V \in \{V^H, V^L\}$, with equal probabilities. We denote the unconditional mean of $\widetilde{V}$ by μ and $\sigma > 0$ represents the fundamental volatility (i.e., standard deviation). The structure of the model and probability distributions of random variables are common knowledge.

Traders. In each trading period, a new risk-neutral trader arrives and may trade at most one unit of the asset (as in Glosten and Milgrom, 1985; Foucault, 1999; Riccó et al., 2020, among others). All traders have a common discount factor $\delta \in (0, 1]$, which is also the same across periods. There are various types of possible investors: rational and liquidity traders. In a given period, the probability that a rational trader arrives at the market is $\lambda > 0$ and for a liquidity trader is $1-\lambda > 0$. Rational traders choose an order submission strategy that maximizes their expected profits conditional on their information sets, denoted by I_t. Rational traders may be either (privately) informed with probability $\pi > 0$ if they have perfect information about the liquidation value of the asset, or (privately) uninformed with probability $1-\pi$ if they only have access to public information. We use $PIN \equiv \lambda\pi$, the probability of informed trading, as a measure of information asymmetry, following Easley and O'Hara (1987) and Easley et al. (1996). An informed trader buys when observing $V = V^H$ (denoted by IH), and sells when

5 For the market shares of these, see Virtu Financial (2022c).

6 Own calculations based on market shares from Virtu Financial (2022c) and information disclosed by broker–dealer and agency dark pools such as Goldman Sachs (2022), J.P. Morgan (2022), Level ATS (2022), and UBS (2022) suggest that at least an additional 24% of other dark pool orders are pegged at the midpoint.

7 For the market shares of these, see the big xyt (2022).

8 For the market shares of these, see the Virtu Financial (2022b) and Virtu Financial (2022a), respectively.

observing $V = V^L$ (denoted by IL).[9] An uninformed trader is a buyer (denoted by UB) with probability $\frac{1}{2}$ or a seller (denoted by US) with probability $\frac{1}{2}$. A liquidity trader negotiates for liquidity or hedging needs, and we assume that buys with probability $\frac{1}{2}$ and sells with probability $\frac{1}{2}$. Uninformed and liquidity traders both have an intrinsic motive to trade (although their motives for purchasing or selling the asset are not explicitly modeled). However, they differ in their immediacy needs: liquidity traders are impatient, while uninformed traders are patient and rationally choose whether to trade, the order type and the venue. Moreover, uninformed traders can learn from public information, so their orders may change from one period to another, while liquidity traders are passive and do not choose among venues as in Ye and Zhu (2020).

Trading venues. The asset may be traded in two venues: an exchange and a dark pool (DP). The exchange is organized as a limit order book (LOB), which is fully transparent (i.e., all of the information is available to all market participants at any point in time), anonymous and a real-time record of previously entered limit orders. A limit order is a type of order to trade an asset at a specific price or better. The LOB matches traders' orders on a price and time priority basis. In our model there are no transaction costs or trading fees. We assume that at the beginning of the game, the initial LOB has at least three prices on the ask (the price that a seller is willing to sell the asset) and bid (the price a buyer is willing to buy the asset) sides of the book: A_1^1, A_1^2, A_1^3, and B_1^1, B_1^2, B_1^3, respectively, such that $V^L \leq B_1^3 < B_1^2 < B_1^1 < \mu < A_1^1 < A_1^2 < A_1^3 \leq V^H$. In addition, prices are placed on a grid and the following relationships hold:

$$A_1^1 = \mu + k_1\tau,\ A_1^2 = \mu + k_2\tau,\ A_1^3 = \mu + k_3\tau,\ V^H = \mu + \kappa\tau,$$
$$B_1^1 = \mu - k_1\tau,\ B_1^2 = \mu - k_2\tau,\ B_1^3 = \mu - k_3\tau,\ V^L = \mu - \kappa\tau,$$

with $1 \leq k_1 < k_2 < k_3 \leq \kappa$, where k_1, k_2, and k_3 are natural numbers, and τ is the tick size (i.e., the minimum price change that traders are allowed to quote over the existing price). Note that the volatility of the asset satisfies $\sigma = \kappa\tau$, with κ being a real number. For simplicity, we assume that the initial depth of the LOB at each bid and ask price is equal to 1.

The way the price grid in the exchange is modeled allows us to start with a full or an almost empty book depending on the parametrization.[10] Thus, for low values of k_1 the limit order book has orders with prices that are close to the midpoint – the mean of the liquidation value of the asset – and therefore, the book is similar to a full book. However, for very high values of k_1 close to κ, the limit order book can be interpreted as an almost empty one.[11] We can interpret $1/k_1$ as a measure of stock liquidity, so the market is very liquid when $k_1 = 1$.

The dark pool is a completely opaque trading venue in the sense that an order submitted to the DP is not observable to anyone besides the trader who submitted it and it does not allow price discovery. As in crossing networks, the DP executes or crosses orders at each trading round, whenever possible, at a price equal to the midpoint of the bid and ask price in the exchange at t: $(A_t^1 + B_t^1)/2$, where A_t^1 and B_t^1 denote the best ask and bid prices at the beginning of trading period t on a time priority basis. In the DP traders obtain a price improvement in relation to the lit market, but face the risk of non-execution and the possibility that, if their orders return to the exchange, the price has moved against them. Note that an order sent to the DP does not change the state of the LOB, and to model the reporting delay of DP trades, we consider that the DP does not report trades until the end of the trading game.

We assume that the probability of execution in the DP in the first trading period is determined by an exogenous order imbalance: the difference between the number of buy and sell orders previously sent to the DP that are unfilled and have not been cancelled. The order imbalance at the beginning of $t = 1$ is denoted by the random variable, $\tilde{z}$, which is realized at $t = 0$ but not observed by traders. Recall that in our model investors might only trade orders of size 1.[12] If the realization of the random variable $\tilde{z}$ is such that $z \geq 1$, then there is an excess of buy orders over sell orders, and a sell order sent to the DP at $t = 1$ executes with certainty; if $z \leq -1$, then there is an excess of sell orders over buy orders, and a buy order sent to the DP at $t = 1$ executes with certainty; and if $-1 < z < 1$, then there is no execution in the DP. Hence, the probability of execution in the DP at $t = 1$ for a rational trader is

$$\theta_1^R = pr_R(\tilde{z} \geq 1) = pr_R(\tilde{z} \leq -1),\ \ R = I, U.$$

Note that we have assumed that $pr_R(\tilde{z} \geq 1) = pr_R(\tilde{z} \leq -1)$, with $R = I, U$ (where I and U indicate informed and uninformed traders, respectively), in order to preserve the symmetry of the model at the beginning of $t = 1$. In particular, due to the symmetry of the model, we restrict $\theta_1^R \leq \frac{1}{2}$, $R = I, U$, where θ_t^I and θ_t^U are the probabilities of execution for an informed and uninformed trader, respectively.[13]

In the second trading period, the probabilities of execution in the DP, θ_2^I and θ_2^U, are endogenous since they depend on the traders' actions at $t = 1$. This is because our objective is to understand how routing orders to the DP in the first trading period affects the rational traders' learning from the exchange. Intuitively, the probability of execution in the DP at $t = 2$ for a rational trader is a weighted average of the probability of execution of a dark order at $t = 1$ of size 1 and the probability of execution of a dark order of size 2. As a special case, when the trader at $t = 1$ sends an order to the LOB, the probability of execution of the dark pool is unchanged (that is, $\theta_2^I = \theta_1^I$ or $\theta_2^U = \theta_1^U$), and it only depends on the exogenous order imbalance. We assume for simplicity that the time period between $t = 1$ and $t = 2$ is so small that there is no change in the exogenous order imbalance in the dark pool.[14]

Notice that, for tractability reasons, we assume that the liquidity supply of the DP is exogenous in the first trading period (as in Ye and Zhu, 2020). We interpret that the potential liquidity in the DP is fragmented and provided by traders outside the model. DP orders come from multiple and diverse sources, such as the order flow of broker–dealers including proprietary flow, market makers, institutional traders and agency algorithms. In Section 2, we reviewed the types of DP in a variety of geographies and noted that broker–dealers and market maker dark pools are both prevalent in the US.[15] Some of these dark pools

[9] For example, an uninformed trader could be a fund manager that rebalances his portfolio for non-informational reasons (see Han et al., 2016), while an informed trader may be a fund manager who uses his connections to acquire information (see Coval and Moskowitz, 2001; Cohen et al., 2008, among others).

[10] Some LOB models assume that the book starts empty (Seppi, 1997; Buti and Rindi, 2013; Buti et al., 2017; Riccó et al., 2020), that is, the only standing limit orders in the initial limit order book are those at "extreme" prices coming from a trading crowd. As Riccó et al. (2020) point out this is a simplification given that, in practice, daily opening limit order books include uncancelled orders from the previous day and new limit orders from opening auctions. In these models, in the first trading period, if an investor wants to trade, he always selects a limit order. By contrast, in an initial non-empty LOB (as in Parlour, 1998; Foucault, 1999) the trader can select any type of order.

[11] Note that we do not need to model a trading crowd willing to provide liquidity at the highest possible prices (as in Seppi, 1997; Parlour, 1998). In their setup, this assumption prevents traders from bidding prices that are too distant from the inside spread. In our framework, since we assume that there are at least three prices previously populated with orders in the LOB and order size is 1, traders will not get to trade against this crowd.

[12] The order should not be interpreted as small. We use the size 1 for tractability reasons.

[13] Throughout the paper, we mention execution risk, which is formally equivalent to $1 - \theta_t^R$.

[14] The detailed derivation of θ_2^I and θ_2^U can be found in Appendix C.

[15] One should note that execution is not guaranteed even if the dark pool is operated by market makers. This is because these have limits on the risk capital.

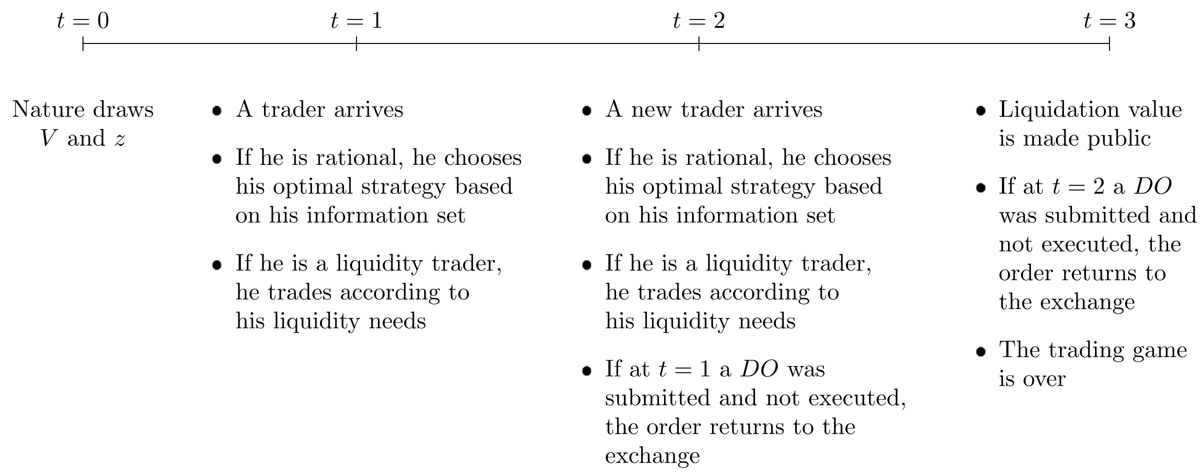

Fig. 1. Timeline of the trading game when traders have access to the DP.

exclusively use the midpoint pricing mechanism, while others have a variety of possible prices, but most include a prevalent order type which is midpoint price peg. For example, Garvey et al. (2016) analyze a DP that resembles our model description and give examples of the DP liquidity providers. The authors use proprietary data for a direct market access broker in the US that allows traders to access the DP through a DP market-maker, and examine a trader's decision to submit to the exchange or to the dark pool.

Traders' strategies. Rational traders simultaneously select whether or not to trade (NT), and if they trade, they choose the trading venue (exchange or DP) and the order type in the exchange. In this venue, traders can submit market orders (MO) or limit orders (LO) to the LOB. A MO is executed immediately at the given best available ask/bid prices, while a LO that improves the current market price may be executed in the next period if a MO of the opposite sign hits the LOB. Thus, LOs may provide better prices than MOs do, but have execution risk. We assume that the DP only admits market orders, DO, which have execution risk. If the order is not executed in the DP at t, then the trader can cancel it or re-route it to the exchange at $t+1$. These traders' orders are called immediate-or-cancel orders (IOC), and this feature allows us to study traders' simultaneous access to the exchange and the dark pool, as in Buti et al. (2017). Consequently, the set of strategies available to a rational trader (both informed and uninformed) is

$$\mathbb{O}_D = \{MO,\ LO,\ DO,\ NT\}, \tag{1}$$

where a B in front of an order type denotes a buy order and a S a sell order. $\Pi^R_{\mathcal{O},t}$ represents the profits of a particular order $\mathcal{O}$ that comes from a rational trader of type R, with $R = I, U$, at time t.

Liquidity traders do not choose between venues and set market orders. Our assumption is similar to Ye and Zhu (2020) who argue that this simplifying assumption is supported by recent empirical evidence that liquidity traders often delegate venue choices to brokers, and the latter often route orders to their own dark pools (Battalio et al., 2016; Anand et al., 2021).

Timing. The sequence of events is illustrated in Fig. 1.

Fig. 2 illustrates the tree of events for the first trading period.[16] The final nodes of the tree include the profits for each of the trading options at $t = 1$.

We can represent our model by a two-period game of incomplete information, and we therefore use the Perfect Bayesian Equilibrium (PBE) concept. In the following, we focus on a symmetric PBE in pure strategies, hereafter, equilibrium. A symmetric equilibrium refers to a situation in which buyers and sellers with the same information (i.e., informed or uninformed) choose the same order type (except the direction of trade).

4. Equilibrium

4.1. Rational traders' expected profits for each type of strategy

For each possible order type, we next examine its characteristics and the associated expected profits for a rational buyer (the sell order profits are analogous). Internet Appendix I derives in detail the expected profits of all traders at all times and for all possible states of the LOB.

Market order (MO): The expected profits of a BMO submitted at date t to the exchange, which executes with certainty and immediately, are

$$\mathbb{E}\left(\Pi^R_{BMO,t}|I_t\right) = \mathbb{E}\left(\widetilde{V}|I_t\right) - A^1_t.$$

Limit order (LO): When a trader chooses to submit a LO to the exchange, it always improves the current price by one tick because: (i) it is never optimal for the trader to improve the price by more than one tick since it reduces his profits; (ii) it is never optimal for the trader to submit a non-improving LO since the order is not executed (due to time priority, the order goes to the end of the queue), and obtains zero profits. The expected profits of a BLO at date t are

$$\mathbb{E}\left(\Pi^R_{BLO,t}|I_t\right) = \delta p^R_{BLO,t}\left(I_t\right)\left(\mathbb{E}\left(\widetilde{V}|I_t\right) - (B^1_t + \tau)\right),$$

where $p^R_{BLO,t}$ is the probability of execution of a BLO submitted by a rational trader of type R at time t.

Dark order (DO): With probability θ^R_t, an order submitted by a rational trader of type R at time t to the DP is executed, and with probability $(1-\theta^R_t)$ it is not executed. Since no new trader arrives in the market at $t = 3$, an order that returns to the exchange from the DP at the end $t = 2$ will be either a MO (we call this dark order $BDO - MO$) or NT (we call this order $BDO - NT$).[17] We denominate the DO as the best of the two: $BDO - MO$ and

[16] We can draw a similar tree of events for the second trading period.

[17] Since the probability of execution of a LO at $t = 3$ is 0, an order will never return to the market as a LO.

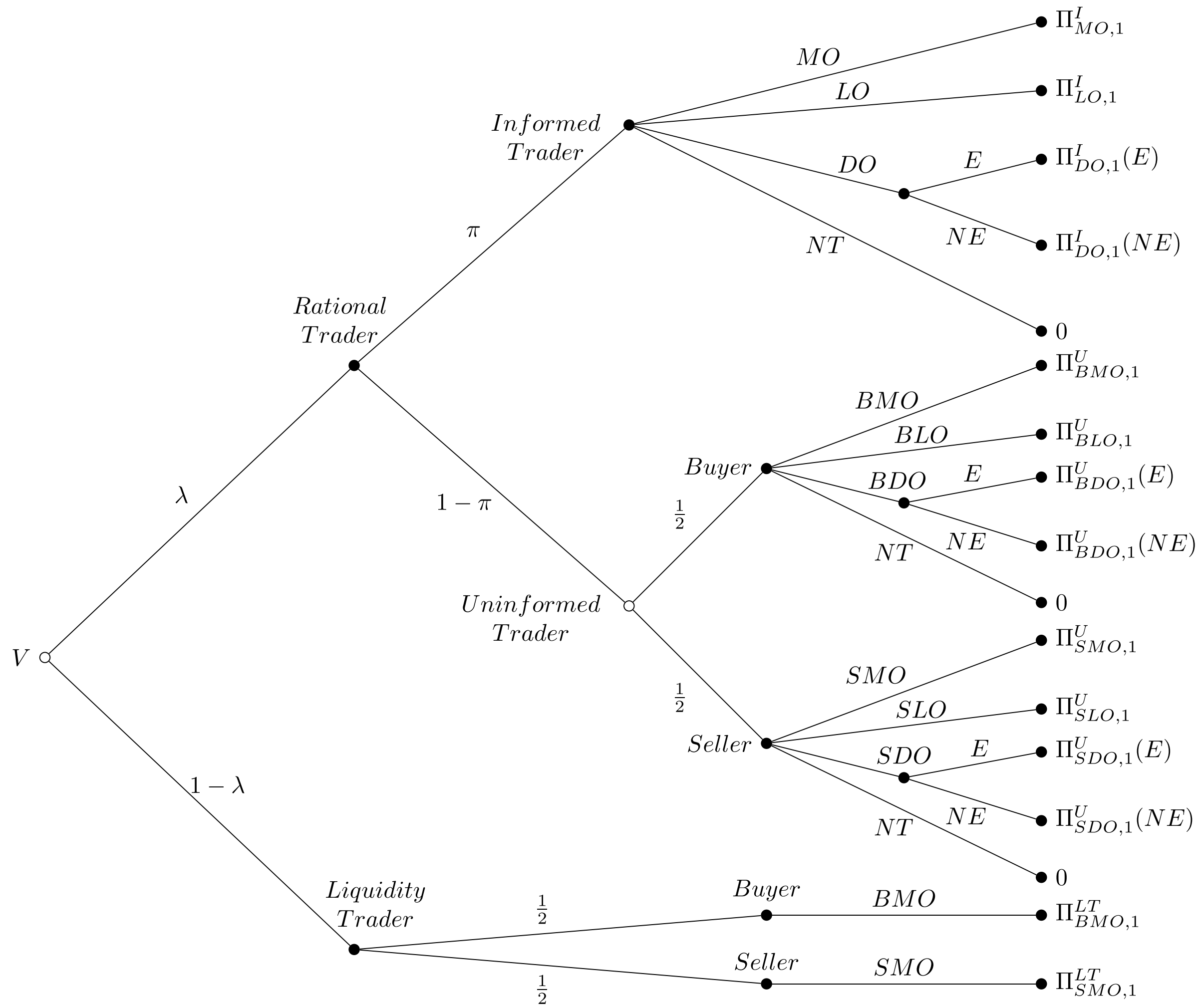

Fig. 2. Tree of events of the first trading period.

$BDO - NT$.[18] Therefore, the expected profits of a BDO submitted at time t are

$$\mathbb{E}\left(\Pi^R_{BDO,t}|I_t\right) = \max\left\{\mathbb{E}\left(\Pi^R_{BDO-MO,t}|I_t\right), \mathbb{E}\left(\Pi^R_{BDO-NT,t}|I_t\right)\right\}$$
$$= \theta^R_t\left(\mathbb{E}\left(\widetilde{V}|I_t\right) - \frac{A^1_t + B^1_t}{2}\right)$$
$$+ (1-\theta^R_t)\delta\max\left\{\mathbb{E}\left(\Pi^R_{BMO,t+1}|I_t\right), 0\right\}.$$

No trade (NT): A trader who refrains from trading at t obtains zero profits, that is,

$$\mathbb{E}\left(\Pi^R_{NT,t}|I_t\right) = 0.$$

In case of equal profits, we assume that a MO dominates both a LO and a DO, and a LO dominates a DO. If the expected profits of a MO are null, then a rational trader refrains from trading.

[18] As we show in the Internet Appendix I, when an informed trader chooses a DO at $t = 1$ and the order is not executed, it is optimal for the informed trader to choose a MO when the order returns to the exchange at the end of the second trading period (i.e., $DO - MO$). In contrast, when an uninformed trader chooses a DO at $t = 1$ and the order is not executed, it is optimal for the uninformed to cancel it at the end of the second trading period (i.e., $DO-NT$). However, at $t = 2$ both types of traders are indifferent between $DO - MO$ and $DO - NT$ since at $t = 3$ the liquidation value is revealed and the profits of both strategies are null.

The final nodes of the tree in Fig. 2 include the profits for each of the trading options at $t = 1$. At the end of the first period, the possible state of the LOB (possible best prices of the LOB) can be: $(A^2_1, B^1_1), (A^1_1, B^2_1), (A^1_1, B^1_1), (A^1_1, B^1_1 + \tau)$, or $(A^1_1 - \tau, B^1_1)$.

4.2. Equilibrium in the single-venue market model

We first consider the single-venue market — where traders can only trade in the exchange. Hence, the set of strategies available to a rational trader is $\mathbb{O}_D \backslash \{DO\}$, that is, a MO, a LO, and NT.

We solve the game backwards. Since the buy and sell sides are separable and symmetric in this model, we focus for exposition on the buy side. The expected profits for the rational traders at $t = 2$ are summarized in Appendix B, Tables B.1 and B.2, while Tables B.4 and B.5 display the expected profits for these traders at $t = 1$. The following lemma presents the informed and uninformed traders' optimal choices at $t = 2$ and $t = 1$.

Lemma 1. *In equilibrium, the following results hold:*

At $t = 2$, an informed trader always submits a MO, while an uninformed trader may submit either a MO or NT, but never chooses a LO.

At $t = 1$, an informed trader may submit either a MO or a LO, but never chooses NT, while an uninformed trader may submit either a LO or NT, but never chooses a MO.

Thus, the candidate strategy profiles at $t = 1$ that can be sustained as a symmetric PBE are:

$$\mathcal{E}_1^{ND} : (BMO,\ SMO,\ BLO,\ SLO), \quad \mathcal{E}_2^{ND} : (BMO,\ SMO,\ NT,\ NT),$$
$$\mathcal{E}_3^{ND} : (BLO,\ SLO,\ BLO,\ SLO), \quad \mathcal{E}_4^{ND} : (BLO,\ SLO,\ NT,\ NT),$$

where the first two components correspond to the strategies of informed traders at $t = 1$ (IH and IL, respectively) and the last two components correspond to the strategies of uninformed traders at $t = 1$ (UB and US, respectively). In what follows the superscript ND indicates that there is no access to the DP, while D indicates that there is access.

The next proposition describes the symmetric PBE of the trading game in the single-venue market.

Proposition 1. *In the single-venue market:*

Case A. *If $k_1 > 1$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, BLO, SLO) & \text{if} \quad \sigma < \kappa^I_{MO-LO}\tau \text{ and } PIN < \psi^U_{LO-NT}, \\ (BLO, SLO, NT, NT) & \text{if} \quad \sigma < \kappa^I_{MO-LO}\tau \text{ and } PIN \geq \psi^U_{LO-NT}, \\ (BMO, SMO, BLO, SLO) & \text{if} \quad \kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN < \psi^U_{LO-NT}, \\ (BMO, SMO, NT, NT) & \text{if} \quad \kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN \geq \psi^U_{LO-NT}, \end{cases}$$

where $\kappa^I_{MO-LO} \equiv \left(k_1 - 1\right) + 2\frac{\delta(k_1-1)(1-\lambda)+1}{2-\delta(1-\lambda)}$, $PIN \equiv \lambda\pi$, *and* $\psi^U_{LO-NT} \equiv \frac{(1-\lambda)(k_1-1)\tau}{\sigma-(k_1-1)\tau}$.

Case B. *If $k_1 = 1$ (the asset is very liquid), then the optimal strategy profile at $t = 1$ is (BMO, SMO, NT, NT).*

For Cases A and B, the optimal strategy of an informed trader at $t = 2$ is to choose a MO for all possible states of the LOB, while an uninformed trader chooses either a MO or NT, depending on his beliefs and market conditions.

Remark 1. Notice that κ^I_{MO-LO} denotes the minimum value of κ such that at $t = 1$ an informed trader chooses a MO instead of a LO, while ψ^U_{LO-NT} represents the minimum value of PIN such that at $t = 1$, an uninformed trader chooses NT instead of a LO.

Remark 2. From the results of Proposition 1, and combining the fundamental volatility and information asymmetry dimensions, we classify assets into Low/High volatility and Low/High PIN, such as *"High–Low"* (i.e., high fundamental volatility– low PIN), leading to four categories of assets. Notice that the optimal strategy profiles at $t = 1$ correspond to the following classification:

"*High-Low*" is $\mathcal{E}_1^{ND}$, "*High-High*" is $\mathcal{E}_2^{ND}$,

"*Low-Low*" is $\mathcal{E}_3^{ND}$, "*Low-High*" is $\mathcal{E}_4^{ND}$.

In the subsequent analysis, we sometimes consider only one of these dimensions in isolation, such as low/high fundamental volatility stocks or low/high PIN.

In the second trading period, an informed trader submits a MO for all states of the LOB. An uninformed trader chooses NT, except if the state of the LOB conveys information about the fundamental value of the asset that could determine to trade using a MO.

In the first trading period, Proposition 1 indicates that when the fundamental asset volatility is sufficiently low (i.e., $\sigma < \kappa^I_{MO-LO}\tau$), it is optimal for an informed trader to supply liquidity (i.e., to place a LO), while the decision of the uninformed trader depends on the severity of adverse selection. Therefore, there are two possible optimal strategy profiles when the asset has low volatility: (BLO, SLO, BLO, SLO) and (BLO, SLO, NT, NT). The first of them occurs in a market with low adverse selection risk (either because the asset's volatility is extremely low, or both the asset's volatility and the PIN are low at the same time). In particular, when the probability of informed trading is low, uninformed traders realize that by placing a LO at the exchange in the first trading period, they are very unlikely to end up trading with informed traders. When adverse selection is sufficiently high (because the asset's volatility is not low and the PIN is high enough) the optimal strategy profile is (BLO, SLO, NT, NT).

By contrast, when the fundamental asset volatility is sufficiently high (i.e., $\sigma \geq \kappa^I_{MO-LO}\tau$), it is optimal for the informed trader to demand liquidity (i.e., to place a MO) in the first trading period. Note that the informational advantage of an informed trader increases with the volatility of the asset (σ). Thus, when the asset's volatility is sufficiently high, an informed trader prefers immediate execution (MO). When σ is not high enough, the informed trader selects a LO because of its price improvement. Furthermore, the uninformed trader's decision depends again on the level of information asymmetry. Consequently, in the case of high volatility, there are two possible optimal strategies: (BMO, SMO, BLO, SLO) and (BMO, SMO, NT, NT). The strategy (BMO, SMO, BLO, SLO) is optimal when the degree of information asymmetry is sufficiently low (i.e., $PIN < \psi^U_{LO-NT}$), while (BMO, SMO, NT, NT) is optimal when the degree of information asymmetry is sufficiently high (i.e., $PIN \geq \psi^U_{LO-NT}$).

Fig. 3 illustrates the optimal strategies in the first trading period of the single-venue market.[19] Numerical simulations show that: (i) the higher the asset's volatility is, the lower the probability of informed trading needs to be for an uninformed trader to choose NT, (ii) the strategy profile (BLO, SLO, NT, NT) is possible only for very specific parameter configurations, such as for $\pi > 0.5$. For this purpose, in the Internet Appendix IV, we show in Figure IV.2 the optimal strategies at $t = 1$ for parameter values that display the four possible equilibria.

The results derived in Proposition 1 are consistent with the previous work by Goettler et al. (2009), who show that informed traders switch from supplying to demanding liquidity when volatility changes from low to high.[20]

The following corollary describes the comparative statics of κ^I_{MO-LO} and ψ^U_{LO-NT} with respect to various market and trader characteristics.

Corollary 1. *Ceteris paribus, κ^I_{MO-LO} increases with δ and k_1, and decreases with λ, while ψ^U_{LO-NT} increases with k_1, and decreases with λ and κ.*

The corollary above implies that for an informed trader at $t = 1$, an increase in the discount factor (δ), a decrease in the liquidity of the asset ($1/k_1$), or an increase in the probability that a liquidity trader arrives at $t = 2$ (ceteris paribus) reduces the relative attractiveness of a MO compared to a LO.[21] Regarding the uninformed trader at $t = 1$, a decrease in the liquidity of the asset, an increase in the probability that a liquidity trader arrives at $t = 2$, or a reduction in the volatility of the asset at $t = 2$ increases the attractiveness of a LO with respect to NT.

Note that according to Corollary 1, the condition $\sigma < \kappa^I_{MO-LO}\tau$ can be satisfied, ceteris paribus, for a low fundamental volatility or low liquidity stock (high k_1), or when rational traders are characterized by low immediacy (high δ) or participate as a relatively small proportion

[19] Figure IV.1 in the Internet Appendix IV shows a similar figure for a liquid market.

[20] Goettler et al. (2009) point out that first as volatility increases, the risk of a LO increases, as they are more likely to be picked-up for trading. Second, as volatility increases, so does the likelihood of finding mispriced orders in the LOB.

[21] The informed trader's profits at $t = 1$ do not depend on the probability that an informed trader arrives in the next trading period, $\lambda\pi$. This is because an informed trader that submits a LO at $t = 1$ knows that this order will not be executed in the next trading period against an order submitted by an informed trader, since an informed trader at $t = 2$ chooses an order of the same sign as the initial order. In addition, an informed trader at $t = 1$ correctly predicts that an uninformed trader at $t = 2$ never submits a MO of the opposite sign as the informed trader at $t = 1$.

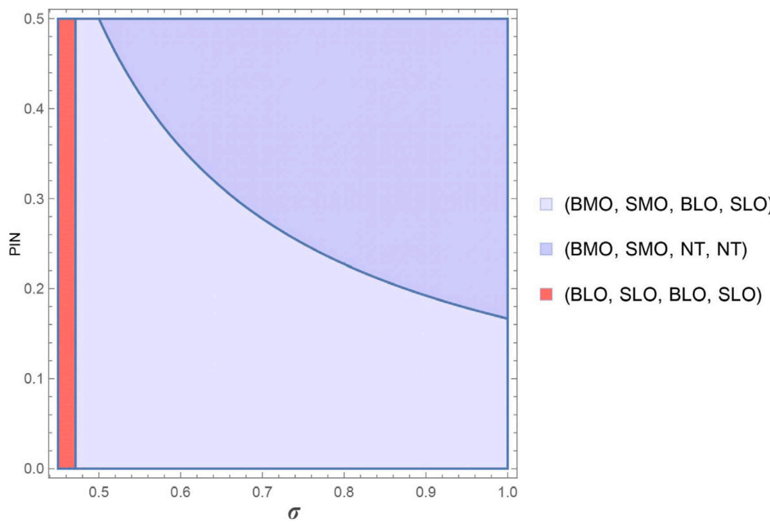

Fig. 3. Optimal strategies at $t = 1$ in the single-venue market. Parameters values: $k_1 = 6$, $\lambda = 0.5$, $\tau = 0.05$, $\delta = 0.95$.

of the market (small λ). To sum up, the characterization of stocks as *"High"* and *"Low"* in terms of liquidity, immediacy, or the proportion of rational traders gives analogous results to the characterization in terms of *"High"* and *"Low"* fundamental volatility. For simplicity, we illustrate our results by discussing them in terms of the fundamental asset volatility, but a similar analysis is possible by studying changes in other stock market and trader characteristics.

4.3. Equilibrium in the two-venue market model

We next consider a two-venue market model in which rational traders have access to both the exchange and the DP. Hence, the orders they can submit are given in (1). A trader's decision to submit an order to the DP depends on price improvement and the probability of execution in this venue at $t = 2$. In particular, we have two cases at $t = 2$, depending whether or not there was a change in the LOB at $t = 1$. When there is a change in the LOB, the probabilities of execution of a dark order are the same as in the previous period. By contrast, if there is no change in the LOB, then these probabilities are revised if the trader believes that this state of the LOB is due to the trader at $t = 1$ submitting a DO (for details, see Lemma C.1 in Appendix C).

As in the previous section, we solve the model backwards. Comparing the expected profits of each of the possible orders for each type of rational trader at $t = 2$ and $t = 1$, Lemma 2 states the strategies that are dominated, and hence, never chosen by a rational player.

Lemma 2. *In equilibrium, the following results hold:*

> *At $t = 2$, an informed trader may submit a MO or a DO, but never chooses a LO or NT, while an uninformed trader may submit either a MO, a DO, or NT, but never chooses a LO.*
> *At $t = 1$, an informed trader may submit either a MO, a LO or a DO, but never chooses NT, while an uninformed trader may submit either a LO or NT, but never chooses a MO or a DO.*

We find that in the second trading period, a LO is never chosen since it is never executed: (a) if the LOB changed in the first period, then no MO arrives at the end of the second trading period, and hence, a LO has zero probability of execution; (b) if the LOB did not change, then a LO can only be executed if an uninformed trader at $t = 1$ chooses a DO, but this cannot occur in equilibrium since the expected profits are null since an uninformed trader at $t = 1$ expects null profits of a DO in any event, executed or not.[22] Moreover, an informed trader at $t = 2$ never chooses NT since it is always dominated by a MO.

In the first trading period, an informed trader never chooses NT since it is always dominated by at least a MO. Moreover, the expected profits of a DO submitted by an informed trader at $t = 1$ are strictly positive (see Table C.3 in Appendix C), and hence, a DO might be optimal for the informed trader at $t = 1$. By contrast, an uninformed trader at $t = 1$ may choose between a LO or NT since the expected profits of a MO are negative and those of a DO are null (see Table C.4 in Appendix C).[23]

Hence, the sustainable candidate strategy profiles at $t = 1$ as a PBE are:

$$\mathcal{E}_1^D : (BMO, SMO, BLO, SLO), \quad \mathcal{E}_2^D : (BMO, SMO, NT, NT),$$
$$\mathcal{E}_3^D : (BLO, SLO, BLO, SLO), \quad \mathcal{E}_4^D : (BLO, SLO, NT, NT),$$
$$\mathcal{E}_5^D : (BDO, SDO, BLO, SLO), \quad \mathcal{E}_6^D : (BDO, SDO, NT, NT).$$

Next, we describe the equilibrium of the trading game in the two-venue market.

Proposition 2. *In the two-venue market:*

Case A. *Suppose $k_1 > 1$. Then, we have the following cases:*

Case A.1 *If $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, BLO, SLO) & \text{if} \quad \theta_1^I \text{ is sufficiently small,} \\ (BDO, SDO, BLO, SLO) & \text{if} \quad \theta_1^I \text{ is sufficiently large.} \end{cases}$$

[22] The uninformed trader at $t = 2$ forms the correct beliefs that if a LO is executed at the end of this trading period, then his counterparty must be the informed trader who arrived at $t = 1$ with probability 1. However, this information reveals to the uninformed buyer (seller) that the value of the asset must be low (high), and hence, the expected profits of a LO are negative.

[23] Note that the mechanism is similar to those in Menkveld et al. (2017) and Brolley (2020): investors weigh each order's execution risk against the price impact. However, in our model, price impact or execution risk are endogenously determined by optimal trading strategies at $t = 1$ and $t = 2$ as traders learn from the LOB.

Case A.2 *If $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \geq \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, NT, NT) & \text{if} \quad \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta^I_1 \text{ is intermediate,} \\ (BDO, SDO, BLO, SLO) & \text{if} \quad \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

Case A.3 *If $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN < \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BMO, SMO, BLO, SLO) & \text{if} \quad \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, BLO, SLO) & \text{if} \quad \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

Case A.4 *If $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$, then the optimal strategies profile at $t = 1$ are:*

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta^I_1 \text{ is intermediate,} \\ (BDO, SDO, BLO, SLO) & \text{if} \quad \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

Case B. *Otherwise, if $k_1 = 1$ (the asset is very liquid), then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

The proof in Appendix C characterizes the threshold values of θ^I_1 for which each strategy profile is optimal at $t = 1$.

For Cases A and B, at $t = 2$ an informed trader chooses either a MO or a DO, while an uninformed trader chooses either a MO, a DO, or NT. Both decisions depend on market conditions, on the probability of execution in the DP and on traders' beliefs.

We start backwards by discussing the second trading period. An informed trader submits a MO (a DO) for all states of the LOB when the probability of execution in the DP is sufficiently low (high) in relation to the price improvement in the new venue. As the probability of execution in the DP increases, an informed buyer replaces a BMO with a BDO in the following order according to the state of the LOB: $(A^2_1, B^1_1), (A^1_1, B^2_1), (A^1_1, B^1_1), (A^1_1, B^1_1 + \tau), (A^1_1 - \tau, B^1_1)$. This occurs because when a BMO was submitted at $t = 1$ and, hence, the best prices in the book are $\left(A^2_1, B^1_1\right)$, the gain from another BMO is the smallest in relation to a BDO even though the probability of execution in the DP is relatively low. However, when a SLO was submitted at $t = 1$ and, hence, the best prices in the book are $(A^1_1 - \tau, B^1_1)$, the gain from a BMO is the largest in relation to that from a BDO despite the fact that the DP probability of execution is relatively high.

For an uninformed trader at $t = 2$, the optimal strategy depends critically on his beliefs about the probability that a MO, a LO or a DO was submitted by an informed trader at $t = 1$. When the state of the LOB contains no information, that is, (A^1_1, B^1_1), then an uninformed trader at $t = 2$ chooses NT since the expected profits of a MO are negative, and the expected profits of a DO are zero because the midpoint price is equal to the unconditional expected liquidation value of the asset. However, an uninformed trader may also choose a MO or a DO in the second trading period if the LOB conveys good news to the trader about the fundamental value of the asset. Note that, in contrast to the first trading period, if the probability of execution in the DP at $t = 2$ is sufficiently high, then an uninformed trader may migrate to the DP.

In the first trading period, Proposition 2 shows that having access to a DP may change the optimal submission strategy profiles for informed and uninformed traders. When the probability of execution in the DP for informed traders at $t = 1$ (i.e., θ^I_1) is sufficiently high, an informed trader switches trading venue, from the exchange to the DP. Otherwise, an informed trader submits the same types of orders to the exchange (MO or LO) as in the single-venue market. The threshold values of the probability of execution in the DP reflect the price improvement and execution trade-off of each order type. In case of execution, the best price is achieved by a LO, followed by a DO, and the worst price is given by a MO. While a LO has execution risk, a MO and a DO do not face execution risk for an informed trader. Note that at $t = 1$, a DO faces no risk of execution since we find that if the DO is not executed in the first trading period, then the informed trader routes it back to the exchange as a MO at the end of the second trading period. However, when this order returns to the exchange, it faces the risk that the price worsened because of the order submitted by the trader that arrives at $t = 2$.

Although an uninformed trader never goes to the DP in the first trading period, as presented in Lemma 2, Proposition 2 shows that the existence of the DP might change the optimal strategy of an uninformed trader when the probability of execution in the DP for an informed trader is high enough, since an uninformed trader may switch from NT to a LO.[24] This is because the high probability of execution in the DP encourages an informed trader at $t = 2$ to trade in the DP rather than in the exchange. Consequently, the adverse selection that the uninformed trader faces at $t = 1$ in the exchange is lower than in the single-venue market, where the informed trader always chooses a MO. This makes the uninformed trade.

Proposition 2 suggests that restricting the informed trader to participate in the DP might harm the uninformed trader. To illustrate this point, notice that Cases A.2 and A.4 of this proposition show that a significant reduction of θ^I_1 might discourage the uninformed trader from participating in the exchange in the first trading period.

Interestingly, our model encompasses both the model of Zhu (2014) and Buti et al. (2017) when the execution probability in the dark is low. Note that when fundamental volatility and the PIN are high, that is, a *"High–High"* stock, the optimal strategy for an informed trader at $t = 1$ is to place a MO as in Zhu (2014). Similarly, when the probability of having an informed trader is very small ($\pi \to 0$), the model is similar to that in Buti et al. (2017), in which there is no asymmetric information. Notice also that when the asset's volatility is low and $\pi \to 0$, traders choose LO at $t = 1$, so the prevailing equilibrium is similar to $\mathcal{E}^D_3$.

Fig. 4 illustrates the optimal strategies at $t = 1$ with respect to the fundamental asset's volatility and information asymmetry for several values of θ^I_1, shown in Panels (a), (b), (c), and (d), respectively.[25] In Panel (a), the graph has the same features as in Fig. 3 since for small values of θ^I_1 there is no migration to the DP. In Panel (b) and (c) we notice that there is a region of parameters in which orders migrate to the DP. Thus, (BLO, SLO, BLO, SLO), (BMO, SMO, BLO, SLO) or (BMO, SMO, NT, NT) prevail when PIN is high and the execution probability in the dark is relatively low. As this probability increases, there is migration to the dark — either $(BDO,\ SDO,\ BLO,\ SLO)$ or $(BDO,\ SDO,\ NT,\ NT)$ prevail, depending on the initial conditions. Notice that as the fundamental volatility increases, the informational advantage of the informed trader becomes higher, and hence, this trader has more incentives to trade immediately. This is because the price improvement of a DO does not compensate the risk of not being executed in the DP at $t = 1$ and returning to the LOB, where the price might have worsened. In Panel (d), we find that the informed trader fully migrates to the DP at $t = 1$, while the uninformed trader decides not to trade whenever the adverse selection he faces is high enough (i.e., when PIN is high enough and the fundamental volatility is not low).[26]

[24] For an example of this switch by the uninformed trader, see Figs. D.1 and D.2 in Appendix D.

[25] Figure IV.3 in the Internet Appendix IV shows similar figures for a liquid market.

[26] In addition to the parameter values defined in the caption of Fig. 4, we assume that the beliefs at $t = 2$ are such that an uninformed buyer (seller) does

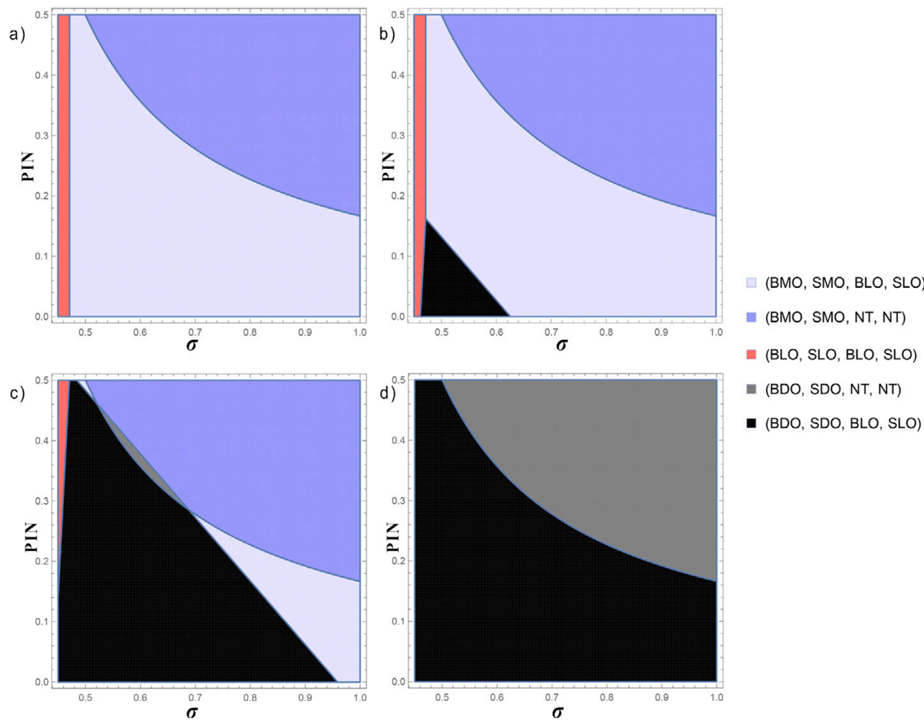

Fig. 4. Optimal strategies at $t = 1$ with DP. Parameters values: $k_1 = 6$, $k_2 = 7$, $\lambda = 0.5$, $\tau = 0.05$, and $\delta = 0.95$. In Panel (a) $\theta_1^I = 0.05$, in Panel (b) $\theta_1^I = 0.0858$, in Panel (c) $\theta_1^I = 0.13$, and in Panel (d) $\theta_1^I = 0.25$.

Few empirical studies have provided estimates of the average probability of execution in dark pools. He and Lepone (2014) compute the execution probability for an exchange operated Australian dark pool and find that the average execution probability (based on volume) is 8.58%. In fact, Fig. 4 Panel (b) uses this value for θ_1^I, and shows that the informed trader will migrate to the dark pool at $t = 1$ in this realistic case. It is an open empirical question whether there are dark pools with higher execution probabilities, especially the ones shown in Fig. 4 Panel (d). Overall, our model's results have highlighted that dark pool execution probabilities that are required to migrate to the DP depend on market conditions.

5. Market performance

In this section we examine how the coexistence of a DP alongside an exchange affects market performance. To do so, we compare several measures of market quality of the two-venue market in relation to the single-venue market: price informativeness, expected inside spread, and rational traders' expected profits. We use the stock categorization with respect to fundamental asset volatility and information asymmetry defined in Section 4, but we can obtain the same empirical implications with respect to initial stock liquidity, traders' immediacy, or rational traders' participation rate.

The signs of the market performance indicators' comparisons may depend on both market and trader characteristics, as well as the trading period, t. Recall that these exogenous characteristics determine the optimal submission strategy for each type of trader both in the single-venue and the two-venue market. In order to show the signs of the comparisons of the different measures in a compact form, we use the following table format:

Market quality parameter	*No Migration to DP at t = 1*				*Migration to DP at t = 1*	
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$ ("*High − Low*")	X				X	
$\mathcal{E}_2^{ND}$ ("*High − High*")		X			X	X
$\mathcal{E}_3^{ND}$ ("*Low − Low*")			X		X	
$\mathcal{E}_4^{ND}$ ("*Low − High*")				X	X	X

The rows of the table show the prevailing equilibria in the single-venue market (Proposition 1), while the columns display the equilibria in the two-venue market (Proposition 2). The cells marked with X show the feasible transitions from the prevailing equilibria in the single-venue market to the ones in the two-venue market. In the empty cells, the comparison is not meaningful (as the transition between these equilibria is not possible). The potential symbols in the comparisons are: " = ", "<", "≤ ", ">", and "≥". The sign " = " means that the market quality measure at t is identical; a "<" ("≤") indicates that the market quality parameter at t corresponding to $\mathcal{E}_i^{ND}$, $i = 1, \ldots, 4$ is lower than (lower or equal to) the market quality parameter at t corresponding to $\mathcal{E}_j^D$, $j = 1, \ldots, 6$; and the reverse is the case for "> " ("≥"). The sign " $\lesseqgtr$" means that the result is ambiguous since it depends on the parameters values.

Remark 3. Proposition 2 and its proof fully characterize the parameter configurations for which there is order migration to the DP at $t = 1$, i.e., when the prevailing equilibrium is either $\mathcal{E}_5^D$or $\mathcal{E}_6^D$.

In what follows, Sections 5.1 and 5.2 present the results of the comparison of market indicators in the first and the second trading period, respectively.

not select a BLO (SLO) when there is no change in the LOB prices, and that an informed buyer (seller) chooses a BMO (SMO) at $t = 2$ when the LOB has not changed. Furthermore, we assume that the probabilities of execution of a DO when the order imbalance is of size 2 submitted by either an informed or uninformed trader at $t = 1$ is equal to zero.

5.1. Market performance in the first trading period

Price informativeness is at the heart of the regulatory debate about whether DPs increase or decrease price discovery. We measure price informativeness in a given trading period t as the reduction in variance of the liquidation value of the asset after observing the set of best ask and bid prices right after finishing the trading process in which a new trader is involved in this trading period.

Proposition 3. *Price informativeness in the first trading period is lower in the two-venue market than in the single-venue market if there has been order migration to the DP; otherwise, price informativeness remains unchanged, as illustrated in the following table:*

Price Informativeness at $t=1$	No Migration to DP at $t=1$				Migration to DP at $t=1$	
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$ ("High – Low")	=				>	
$\mathcal{E}_2^{ND}$ ("High – High")		=			>	>
$\mathcal{E}_3^{ND}$ ("Low – Low")			=		>	
$\mathcal{E}_4^{ND}$ ("Low – High")				=	>	>

As the previous table shows, in the first trading period, when there is no migration to the DP (i.e., when the prevailing equilibrium in the two-venue market is $\mathcal{E}_i^D$, $i = 1, \ldots, 4$), price informativeness at $t = 1$ remains the same as in the single-venue market. However, when there is migration to the DP, price informativeness is lower. This decrease in price informativeness is a direct consequence of the order flow segmentation at $t = 1$ since, in this period, the DP is only attractive to informed traders. Our result may explain the existing empirical results of Hendershott and Jones (2005), Comerton-Forde and Putniņš (2015) (when the proportion of dark trading is above 10%), Hatheway et al. (2017), and Brogaard and Pan (2021). With regards to the order flow segmentation, Naes and Odegaard (2006) find that there is informational content in crossing network trades, Nimalendran and Ray (2014) find that informed traders strategically use both crossing networks and exchanges, and Garvey et al. (2016) provide evidence that traders who use the DP more often are better forecasters of the future price direction.

The next proposition shows how access to a DP affects market liquidity, measured by the ex-ante expected inside spread in the exchange, denoted by $\mathbb{E}_0(S_1)$, where the inside spread is the difference between the best ask and the bid prices at the end of the first trading period.

Proposition 4. *In the end of the first trading period, the expected inside spread is lower (higher) in the two-venue market for high (low) fundamental volatility stocks when there is order migration to the DP; otherwise, it remains unchanged, as illustrated in the following table:*

$\mathbb{E}_0(S_1)$	No Migration to DP at $t=1$				Migration to DP at $t=1$	
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$ ("High – Low")	=				>	
$\mathcal{E}_2^{ND}$ ("High – High")		=			>	>
$\mathcal{E}_3^{ND}$ ("Low – Low")			=		<	
$\mathcal{E}_4^{ND}$ ("Low – High")				=	<	<

In the first trading period, for high fundamental volatility stocks (for which the prevailing equilibrium is in the single-venue market either $\mathcal{E}_1^{ND}$ or $\mathcal{E}_2^{ND}$), the existence of a DP makes the informed trader switch from a MO to a DO, and therefore, the expected inside spread decreases, regardless of the behavior of the uninformed trader. We can explain this result by noting that the switch from a MO to a DO reduces the inside spread because the trader does not consume liquidity in the exchange.[27] Therefore, for high volatility stocks our results may explain the empirical studies that show that DP trading increases market liquidity (Gresse, 2006; Buti et al., 2022; Ready, 2014).[28]

In contrast, for low fundamental volatility stocks (for which the prevailing equilibrium in the single-venue market is either $\mathcal{E}_3^{ND}$ or $\mathcal{E}_4^{ND}$), the switch from a LO to a DO increases the inside spread. This is because the informed trader does not supply liquidity to the exchange. At the same time, the switch from NT to a LO by an uninformed trader reduces the inside spread, as the uninformed trader does supply liquidity to the exchange. Hence, for the *"Low-High"* stocks, a potential ambiguity arises when we compare $\mathcal{E}_4^{ND}$ to $\mathcal{E}_5^D$. However, note that $\mathcal{E}_4^{ND}$ prevails only if the probability that an informed trader arrives is sufficiently high $\left(\pi > \frac{1}{2}\right)$. Hence, the effect of the informed trader on the inside spread dominates the effect of the uninformed trader, and therefore, the expected inside spread in the two-venue market is unequivocally larger than in the single-venue market. These results for low volatility stocks may explain the empirical studies that show that the existence of a DP decreases market liquidity (Nimalendran and Ray, 2014; Weaver, 2014; Kwan et al., 2015; Degryse et al., 2015; Hatheway et al., 2017). Finally, when there is no migration to the DP, the spread stays the same. Foley and Putniņš (2016) show that midpoint dark trading in the Canadian market does not benefit or harm market liquidity, and Gresse (2017) shows that dark trading is not harmful to any dimension of market liquidity.

In what follows, we compare the (unconditional) expected profits of rational traders at $t = 1$ in the two-venue market in relation to the single-venue market.

Proposition 5. *In the first trading period, the existence of a DP alongside the exchange increases or leaves the same informed traders' expected profits and it has the following effects on uninformed traders' expected profits:*

$\mathbb{E}_0(\Pi_{1,U})$	No Migration to DP at $t=1$				Migration to DP at $t=1$	
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$ ("High – Low")	=				≤	
$\mathcal{E}_2^{ND}$ ("High – High")		=			<	=
$\mathcal{E}_3^{ND}$ ("Low – Low")			=		≤	
$\mathcal{E}_4^{ND}$ ("Low – High")				=	<	=

An informed trader strictly increases his expected profits when choosing a DO since the price improvement obtained by submitting a DO outweighs the execution risk in the DP. An uninformed trader has larger expected profits in the two-venue market relative to the single-venue market even when he does not go to the DP at $t = 1$, but the market conditions are such that informed trader goes to the DP and the prevailing equilibrium is $\mathcal{E}_5^D$. This is because the migration of the informed trader's orders to the DP reduces adverse selection in the LOB. Note that in this case the probability that an uninformed trader faces an informed trader is smaller in the two-venue market and this might encourage to uninformed traders to set a LO. Therefore, the uninformed trader's expected profits are higher or equal in the two-venue market compared to the single-venue market.

5.2. Market performance in the second trading period

In the previous subsection, we have focused on the effects of the coexistence of a DP alongside an exchange in the first trading period. Interestingly, traders' decisions in the second period depend on the decisions of traders that arrived at the market at $t = 1$ (and reflected in the state of the limit order book at the beginning of $t = 2$). Note that since the game ends right after the second period, the execution

[27] Note that in cases $A.3$ and $A.4$ in Proposition 2 the uninformed trader either does not change his order type or changes from NT to LO. This last change also reduces the inside spread.

[28] Our results for high fundamental volatility stocks are in line with the theoretical conjecture by Buti et al. (2017) that dark trading would not necessarily cause a wider spread even under asymmetric information. However, our results differ for low fundamental volatility stocks.

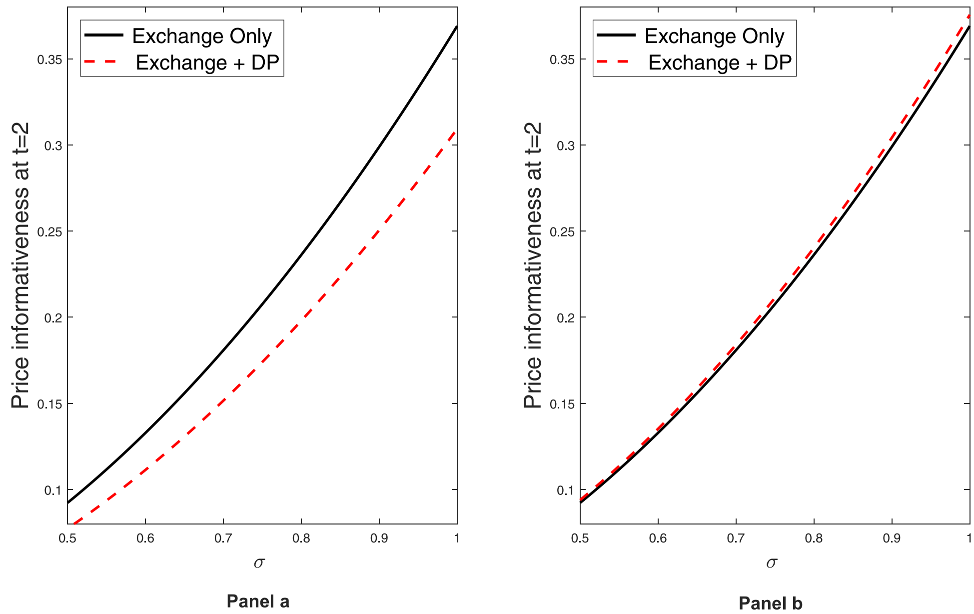

Fig. 5. Price informativeness at $t = 2$. Parameters values: $k_1 = 5$, $k_2 = 6$, $\lambda = 0.75$, $\pi = 0.5$, $\tau = 0.05$, and $\delta = 0.9$. In Panel (a) the values of θ_2^I and θ_2^U are such that only the informed trader goes to the DP at $t = 2$. In Panel (b) the values of θ_2^I and θ_2^U are such that only the uninformed trader goes to the DP at $t = 2$.

probability of a LO at $t = 2$ is zero, and as a result, we cannot disentangle the end of game effects from the (endogenous) effects brought about by $t = 2$ optimal decisions.

Proposition 6. *The coexistence of a DP with an exchange has the following effects on price informativeness at the end of the second trading period:*

Price Informativeness at $t = 2$	*No Migration to DP at $t = 1$*				*Migration to DP at $t = 1$*	
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$ (*"High – Low"*)	$\geq$				$>$	
$\mathcal{E}_2^{ND}$ (*"High – High"*)		$\lesseqgtr$			$>$	$>$
$\mathcal{E}_3^{ND}$ (*"Low – Low"*)			$\lesseqgtr$		$>$	
$\mathcal{E}_4^{ND}$ (*"Low – High"*)				$\lesseqgtr$	$>$	$>$

This table shows the effects of the existence of the DP on price informativeness in the second period depend crucially on the initial market conditions in the lit market and the execution probability in the DP. When market conditions are such that there is migration to the DP at $t = 1$, the price informativeness is always lower in the two-venue market. In contrast, when there is no migration to the DP at $t = 1$, we might have that price informativeness can be both higher or lower in the two-venue market than in the single-venue market, depending on how order flow gets segmented. In the case of a stock for which volatility is high and PIN is low (the equilibrium *"High-Low"*) we show that the price informativeness is lower in the two-venue market than in the single-venue market. However, in the case of a *"High-High"* stock for example, the initial market conditions determine whether there is or not order flow segmentation in the second period (since in this period both the informed and uninformed trader might submit orders to the DP if conditions are favorable). The change in the DP's order attractiveness for uninformed traders between the first and the second trading period brings about in this case differences in how the coexistence of the DP with the LOB affects price informativeness. In this case when the initial execution probability in the dark pool is low, in equilibrium at $t = 1$ an informed trader chooses a MO and an uninformed trader NT both in the single-venue and two-venue market; that is, $\mathcal{E}_2^{ND}$ and $\mathcal{E}_2^D$. When there is no change in traders' behavior at $t = 2$, then price informativeness stays the same. However, if at $t = 2$ informed and uninformed traders choose different trading venues, then we have contrasting results regarding price informativeness in the second trading period. Thus, if there is segmentation of the order flow such that the informed trader goes to the DP, while the uninformed trader remains in the exchange, then we expect a reduction in price informativeness analogous to the first trading period (see Fig. 5, Panel a). By contrast, when there is segmentation of the order flow but the informed trader stays in the exchange and the uninformed trader migrates to the DP, then we expect an increase in price informativeness in the second trading period (see Fig. 5, Panel b). Note that in this example, the stock market and trader characteristics are the same for Panels (a) and (b), and are such that equilibria $\mathcal{E}_2^{ND}$ and $\mathcal{E}_2^D$ arise. The key variables that determine the traders' behavior and lead to an increase or a decrease in price informativeness in this example are the execution probabilities in the DP at $t = 2$. However, stock market and trader characteristics are also important in the magnitude of this change (for instance, price informativeness increases with fundamental volatility, as does the increase/decrease in price informativeness due to the coexistence of the DP with the exchange).

The next proposition shows how access to a DP affects market liquidity in the second trading period, $\mathbb{E}_0(S_2)$.

Proposition 7. *The existence of a DP alongside the exchange has the following effects on ex-ante expected spreads at the end of the second trading period:*

$\mathbb{E}_0(S_2)$	*No Migration to DP at $t = 1$*				*Migration to DP at $t = 1$*	
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$ (*"High – Low"*)	$\geq$				$>$	
$\mathcal{E}_2^{ND}$ (*"High – High"*)		$\geq$			$>$	$>$
$\mathcal{E}_3^{ND}$ (*"Low – Low"*)			$\geq$		$\lesseqgtr$	
$\mathcal{E}_4^{ND}$ (*"Low – High"*)				$\geq$	$>$	$\lesseqgtr$

With respect to market liquidity in the second trading period, we note that at the beginning of the second trading period, we could have different spreads depending on whether the DP is available or not. Thus, for high volatility stocks, we expect that the inside spread at the beginning of $t = 2$ in the two-venue market to be lower or to stay the same (see Proposition 4). In these cases, having access to the DP unambiguously reduces the ex-ante expected inside spread. This is because at $t = 2$, an informed trader might switch from a MO to a DO, and an uninformed trader from a MO or NT to a DO, which reduces the expected inside spread. However, we can obtain different results for low volatility stocks for which we expect, in equilibrium, a higher inside spread at the beginning of the second trading period in the two-venue market in case of order migration to DP at $t = 1$. Thus, the possibility of submitting a DO instead of a MO in the second trading period might reduce the inside spread in the two-venue market, which goes in the opposite direction to the one obtained in the first trading period.

The next proposition shows how access to a DP affects rational traders' *expected profits* in the second trading period.

Proposition 8. *At $t = 2$, the existence of a DP alongside the exchange increases or leaves the same the informed traders' expected profits and it has the following effects on uninformed traders' expected profits:*

$\mathbb{E}_0(\Pi_{2,U})$	*No Migration to DP at* $t = 1$				*Migration to DP at* $t = 1$	
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$ (*"High – Low"*)	$<$				$<$	
$\mathcal{E}_2^{ND}$ (*"High – High"*)		$\leq$			$\lesseqgtr$	$\lesseqgtr$
$\mathcal{E}_3^{ND}$ (*"Low – Low"*)			$<$		$\lesseqgtr$	
$\mathcal{E}_4^{ND}$ (*"Low – High"*)				$<$	$\lesseqgtr$	$\lesseqgtr$

With respect to expected profits in the second trading period, the informed trader's expected profits are not lower in the two-venue market. However, the changes in the uninformed trader's expected profits at $t = 2$ (denoted by $\mathbb{E}_0(\Pi_{2,U})$) depend on the market conditions. A priori one would expect that when uninformed traders have the opportunity to trade in the DP their welfare increases. But from Proposition 3, we know that uninformed traders can extract less information from the book in the two-venue market if there is order migration, which makes them worse off. For *"High-Low"* stocks the uninformed trader's expected profits are always larger in the two-venue market. This is because in this market structure uninformed traders go to the DP in some states of the book, while in the single-venue market, uninformed traders do not trade. For the other types of stocks, the uninformed trader's expected profit are higher except when the PIN is high and the execution probability in DP of uninformed, θ_2^U, is small. In this situation, it is not beneficial for an uninformed trader to leave the exchange and migrate to the DP. However, if the execution probability in DP, θ_2^U, is high then the expected profits of the uninformed trader are higher in the two-venue market.

6. Concluding remarks

This paper examines the impact of an opaque dark pool that competes with a transparent exchange organized as a limit order book in a model with asymmetric information about the liquidation value of the asset. We find that the effects of this competition on price informativeness and market performance depend critically on the stock categorization in terms of high/low fundamental volatility and high/low information asymmetry as these factors are determinant when selecting the venue and the type of order. As a result, regulators should take into account the market conditions in the implementation of policies that aim to curb dark trading.

The existing empirical research often gives conflicting results on the effects of the presence of a DP alongside an exchange. Studies differ in their research questions, the type of data, and regulatory environments. Thus, most of these empirical studies suggest that the discrepancies are driven by differences in the market structure and financial regulations. Interestingly, our analysis predicts that the coexistence of the two venues may have both negative and positive effects on the market performance of the LOB, even if the market structure and regulatory environment are exactly the same. As our previous analysis shows, stock and trader characteristics affect the optimal order submission strategies, and in turn, these have implications for market quality and traders' profits.

Future work could extend our theoretical model in different ways: modeling asymmetrical bid and ask prices in the limit order book, or modeling the execution probability in the dark pool as an increasing function of fundamental volatility (as the empirical literature suggests). Additionally, our results call for the development of applied work studying the effects of asymmetric information in the competition between trading venues with different degrees of transparency on market quality and traders' profits.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendices

Appendix A is a summary of the notation used; Appendix B shows the proofs related to the single venue market model (Lemma 1 and Proposition 1); Appendix C includes the proofs related to the two-venue market model (Lemma 2, Lemma C.1, Lemma C.2 and Proposition 2); and Appendix D includes two figures which complement Fig. 4. The rest of the proofs and further details of the calculations underlying our results can be found in the Internet Appendices.

Appendix A. Notation summary

This appendix summarizes the key notations used in the paper.

Types of Traders	
Type	Definition
R	Rational trader, $R \in \{I, U\}$
I (IH/IL)	Informed trader (who observes a high/low liquidation value)
U (UB/US)	Uninformed trader (who buys/sells)
LT	Liquidity trader
Types of Orders	
Type	Definition
MO (BMO/SMO)	Market order (Buy/Sell market order)
LO (BLO/SLO)	Limit order (Buy/Sell limit order)
DO (BDO/SDO)	Dark order (Buy/Sell dark order)
NT	No trade
Exogenous Variables	
Parameters	Definition
$\widetilde{V}$	Liquidation value of the asset, which may take two values $V \in \{V^H, V^L\}$
μ and σ	The unconditional mean and volatility of the liquidation value $\widetilde{V}$
A_1^p, B_1^p	Ask and bid prices at time $t = 1$ and position p
τ	Tick size
k_p	A natural number such that $A_1^p = \mu + k_p\tau$ $(B_1^p = \mu - k_p\tau)$
κ	A real number such that $\sigma = \kappa\tau$
λ	Probability that a rational trader arrives to the market
π	Probability that a rational trader is informed
PIN	Probability that an informed trader arrives to the market, which is equal to $\pi\lambda$
δ	Discount factor (immediacy) of rational traders
$\widetilde{z}$	Order imbalance in the dark pool at the beginning of the first trading period
θ_1^R	Probability of execution of a DO at $t = 1$ for a rational trader

Endogenous Variables	
Variable	Definition
A_2^p, B_2^p	Ask and bid prices at time $t = 2$ and position p
θ_2^R	Probability of execution of a DO at $t = 2$ for a rational trader, R
$\Pi_{O,t}^R$	Profit for a trader of type R using order O at date t
$\kappa_{MO-LO}^I\tau$	Volatility threshold for informed trader's decision between MO and LO
ψ_{LO-NT}^U	PIN threshold for uninformed trader's decision between LO and NT

LOB denotes the limit order book, DP denotes the dark pool, ND denotes the single-venue market, and D denotes the two-venue market.

Appendix B. Single-venue market model

In the next definitions, we introduce some notations that will be used in the proofs included in Appendix B.

Definition B.1. Let us define Ω_o and Γ_o as the probability that an informed trader and uninformed trader at $t = 1$ choose an order $\mathcal{O} \in \mathbb{O}_{ND}$, where $o = 0$ corresponds to a NT order; $o = 1$ to a MO; $o = 2$ to a LO; and such that $\sum_{o=0}^{2} \Omega_o = 1$ and $\sum_{o=0}^{2} \Gamma_o = 1$.

Proof of Lemma 1. We solve the game backwards. At $t = 2$, the expected profits for an informed and uninformed trader are given in Table B.1 and Table B.2, respectively.

Table B.1
Expected profits of an informed buyer (IH) and an informed seller (IL) at $t = 2$ when traders do not have access to the dark pool.

State of the book	IH			IL		
	BMO	BLO	NT	SMO	SLO	NT
(A_1^1, B_1^1)	$(\kappa - k_1)\,\tau$	0	0	$(\kappa - k_1)\,\tau$	0	0
(A_1^2, B_1^1)	$(\kappa - k_2)\,\tau$	0	0	$(\kappa - k_1)\,\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(\kappa - k_1)\,\tau$	0	0	$(\kappa - k_1 + 1)\,\tau$	0	0
(A_1^1, B_1^2)	$(\kappa - k_1)\,\tau$	0	0	$(\kappa - k_2)\,\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$(\kappa - k_1 + 1)\,\tau$	0	0	$(\kappa - k_1)\,\tau$	0	0

Table B.2
Expected profits of an uninformed buyer (UB) and an uninformed seller (US) at $t = 2$ when traders do not have access to the dark pool.

State of the book	UB			US		
	BMO	BLO	NT	SMO	SLO	NT
(A_1^1, B_1^1)	$-k_1\tau$	0	0	$-k_1\tau$	0	0
(A_1^2, B_1^1)	$(X\kappa - k_2)\,\tau$	0	0	$-(k_1 + X\kappa)\,\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(Y\kappa - k_1)\,\tau$	0	0	$-(k_1 - 1 + Y\kappa)\,\tau$	0	0
(A_1^1, B_1^2)	$-(X\kappa + k_1)\,\tau$	0	0	$(X\kappa - k_2)\,\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-(Y\kappa + k_1 - 1)\,\tau$	0	0	$(Y\kappa - k_1)\,\tau$	0	0

Note that at $t = 2$ the expected profits of each strategy depend on the state of the LOB (which depends on the chosen strategy at $t = 1$). Uninformed traders at $t = 2$ form beliefs about the strategies and type of player at $t = 1$. Thus, we define the uninformed traders' belief at $t = 2$ about the probability that the MO (observed in the LOB) was submitted by an informed trader as

$$X = \frac{\lambda\pi\Omega_1}{1 - \lambda + \lambda\pi\Omega_1 + \lambda(1 - \pi)\Gamma_1}. \tag{B.1}$$

Similarly, we define the uninformed traders' belief at $t = 2$ about the probability that the LO (observed in the LOB) was submitted by an informed trader as

$$Y = \frac{\pi\Omega_2}{\pi\Omega_2 + (1 - \pi)\Gamma_2}. \tag{B.2}$$

By comparing the expected profits of an informed trader at $t = 2$ we obtain that the informed trader always submits a MO. Similarly, we compare the profits of an uninformed trader and see that he never chooses to submit a LO. His choice between a MO or NT depends on the uninformed trader beliefs that the order placed at $t = 1$ that he observes in the book comes from an informed trader, as it can be seen in Table B.3.

Table B.3
Optimal trading strategies of an uniformed buyer (UB) and uninformed seller (US) at $t = 2$ when traders do not have access to the dark pool.

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } X\kappa > k_2 \\ NT & \text{if } X\kappa \le k_2 \end{cases}$	NT
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \le k_1 \end{cases}$	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } X\kappa > k_2 \\ NT & \text{if } X\kappa \le k_2 \end{cases}$
$(A_1^1 - \tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \le k_1 \end{cases}$

At $t = 1$, the expected profits of an informed and uninformed trader are presented in Table B.4 and Table B.5, respectively.

Table B.4
Expected profits of an informed buyer (IH) and informed seller (IL) at $t = 1$ when traders do not have access to the dark pool.

IH	IL	Expected Profits
BMO	SMO	$(\kappa - k_1)\,\tau$
BLO	SLO	$\delta\frac{1-\lambda}{2}(\kappa + k_1 - 1)\,\tau$
NT	NT	0

Table B.5
Expected profits of an uninformed buyer (UB) and uninformed seller (US) at $t = 1$ when traders do not have access to the dark pool.

UB	US	Expected Profits
BMO	SMO	$-k_1\tau$
BLO	SLO	$\frac{\delta}{2}\left((1 - \lambda + \lambda\pi)(k_1 - 1) - \lambda\pi\kappa\right)\tau$
NT	NT	0

It can be easily seen that at $t = 1$ the informed trader never chooses NT, while the uninformed never chooses a MO.

Proof of Proposition 1. We follow the steps outlined in Internet Appendix II to check if a particular strategy profile constitutes a PBE. Because of the symmetry of the model, without any loss of generality, at $t = 1$ we focus on buyers. We distinguish two cases: Case A ($k_1 > 1$) and Case B ($k_1 = 1$).

Case A. We present the full proof for one of the possible strategy profile at $t = 1$ that yields an equilibrium. The proofs of all the other 3 equilibria are sketched here and can be obtained on request from the authors.

$\mathcal{E}_1^{ND}$: (BMO, SMO, BLO, SLO)

First step. In this case $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$, $\Gamma_0 = 0$, $\Gamma_1 = 0$, and $\Gamma_2 = 1$.

Second step. Using Bayes' rule we obtain that $X^{1,ND} = \frac{\lambda\pi}{1 - \lambda + \lambda\pi}$ and $Y^{1,ND} = 0$.

Third step. Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of the uninformed trader is given in Table B.6.

Fourth step. Given the optimal response of traders at $t = 2$, we find the optimal action for all rational traders at $t = 1$.

Table B.6
Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$	NT
$(A_1^1, B_1^1+\tau)$	NT	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$
$(A_1^1-\tau, B_1^1)$	NT	NT

Table B.7
Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, NT, NT).

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$

Informed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$\kappa - k_1 \geq \delta\frac{1-\lambda}{2}\left(\kappa + k_1 - 1\right). \tag{B.3}$$

Uninformed traders at $t = 1$ have no incentives to deviate from the prescribed strategy if and only if

$$(1-\lambda)(k_1-1) - \lambda\pi\left(\kappa - \left(k_1 - 1\right)\right) > 0. \tag{B.4}$$

Fifth step. Nobody at $t = 1$ has unilateral incentives to deviate from (BMO, SMO, BLO, SLO) when both conditions (B.3) and (B.4) are satisfied, and these conditions can be rewritten as

$$\kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN < \psi^U_{LO-NT},$$

where the expressions of ψ^U_{LO-NT} and κ^I_{MO-LO} are given in the statement of this proposition.

Finally, combining Table B.6 and Expression (B.4), it follows that an uninformed trader always selects NT at $t = 2$.

$\mathcal{E}_2^{ND}$: (BMO, SMO, NT, NT)

Following the same procedure as above and noting that $X^{2,ND} = \frac{\lambda\pi}{1-\lambda+\lambda\pi}$ and $Y^{2,ND}$ is undetermined $Y^{2,ND} \in [0,1]$, we obtain that no trader at $t = 1$ has unilateral incentives to deviate in $\mathcal{E}_2^{ND}$ whenever:

$$\kappa - k_1 \geq \delta\frac{1-\lambda}{2}\left(\kappa + k_1 - 1\right) \text{ and} \tag{B.5}$$

$$0 \geq (1-\lambda)(k_1-1) - \lambda\pi\left(\kappa - \left(k_1 - 1\right)\right), \tag{B.6}$$

which can be rewritten as $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$. Finally, in Table B.7 we include the moves that are in the equilibrium path at $t = 2$ for an uninformed trader, taking into account that (BMO, SMO, NT, NT) is the strategy profile chosen at $t = 1$.

$\mathcal{E}_3^{ND}$: (BLO, SLO, BLO, SLO)

Following the same procedure as above and noting that $X^{3,ND} = 0$ and $Y^{3,ND} = \pi$, we obtain that no trader at $t = 1$ has unilateral incentives to deviate in $\mathcal{E}_3^{ND}$ whenever:

$$\delta\frac{1-\lambda}{2}\left(\kappa + k_1 - 1\right) > \kappa - k_1 \text{ and} \tag{B.7}$$

Table B.8
Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, BLO, SLO).

State of the book	UB	US
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1+\tau)$	$\begin{cases} MO & \text{if } \pi\kappa > k_1 \\ NT & \text{if } \pi\kappa \leq k_1 \end{cases}$	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1-\tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } \pi\kappa > k_1 \\ NT & \text{if } \pi\kappa \leq k_1 \end{cases}$

Table B.9
Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1+\tau)$	MO	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1-\tau, B_1^1)$	NT	MO

$$(1-\lambda)(k_1-1) - \lambda\pi\left(\kappa - \left(k_1 - 1\right)\right) > 0, \tag{B.8}$$

which can be rewritten as $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$.

Finally, in Table B.8 we include the moves that are in the equilibrium path at $t = 2$ for an uninformed trader, taking into account the conditions that must be satisfied if (BLO, SLO, BLO, SLO) is the strategy profile chosen at $t = 1$.

$\mathcal{E}_4^{ND}$: (BLO, SLO, NT, NT)

Following the same procedure as above and noting that $X^{4,ND} = 0$ and $Y^{4,ND} = 1$, we obtain that no trader at $t = 1$ has unilateral incentives to deviate in $\mathcal{E}_4^{ND}$ whenever:

$$\delta\frac{1-\lambda}{2}\left(\kappa + k_1 - 1\right) > \kappa - k_1 \text{ and} \tag{B.9}$$

$$0 \geq (1-\lambda)(k_1-1) - \lambda\pi\left(\kappa - \left(k_1 - 1\right)\right), \tag{B.10}$$

which can be rewritten as $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \geq \psi^U_{LO-NT}$.

Finally, in Table B.9 we include the moves that are in the equilibrium path at $t = 2$ for an uninformed trader, taking into account the conditions that must be satisfied if (BLO, SLO, NT, NT) is the strategy profile chosen at $t = 1$.

Case B. We have to replace $k_1 = 1$ in the proof of Case A. It should only be noted that when $k_1 = 1$ the conditions (B.4) and (B.8) are never satisfied and, therefore, the strategy profiles at $t = 1$ (BMO, SMO, BLO, SLO) and (BLO, SLO, BLO, SLO) cannot be part of an equilibrium of the game. By contrast, when $k_1 = 1$, the conditions (B.6) and (B.10) always hold. However, the condition (B.9) is never satisfied when $k_1 = 1$, and therefore, the strategy profile (BLO, SLO, NT, NT) cannot be either part of an equilibrium of the game.

Appendix C. Two-venue market model

In the next definitions, we introduce some notations that will be used in the proofs included in Appendix C.

Definition C.1. Let us define Ω_o and Γ_o as the probability that an informed trader and uninformed trader at $t = 1$ choose an order $\mathcal{O} \in \mathbb{O}_D$, where $o = 0$ corresponds to a NT order; $o = 1$ to a MO; $o = 2$ to a LO; $o = 3$ to a DO; and such that $\sum_{o=0}^{3}\Omega_o = 1$ and $\sum_{o=0}^{3}\Gamma_o = 1$.

Definition C.2. We define as $\mathbb{B}_1$ the set of all possible states of the LOB at the end of the first trading period and by $B_1 \in \mathbb{B}_1$ a possible state of the book (see Internet Appendix I for a full definition). The state of the book $\mathcal{B}_1 = \emptyset$ is the state when the best prices at the end of the first trading period in the book are (A_1^1, B_1^1).

Lemma C.1 (*(Derivation of the Probabilities of Execution of a Dark Order at $t = 1$ and $t = 2$)*). *The probabilities of execution in the dark pool at $t = 1$ and $t = 2$ depend on the order imbalance $\tilde{z}$, market conditions, and changes in the LOB.*

Proof of Lemma C.1. Let $\tilde{z}$ denote the order imbalance of the DP at the beginning of $t = 1$. This random variable is realized at $t = 0$, but not observed by investors. In order to preserve the symmetry of the model it is assumed that this random variable satisfies $\theta_1^{R,1} = pr_R\left(\tilde{z} \ge 1\right) = pr_R\left(\tilde{z} \le -1\right)$ and $\theta_1^{R,2} = pr_R\left(\tilde{z} \ge 2\right) = pr_R\left(\tilde{z} \le -2\right)$, $R = I, U$. Therefore, $\theta_1^{I,1}$ denotes the probability of execution of a dark order of size 1 for an informed trader at $t = 1$, while $\theta_1^{I,2}$ denotes the probability of execution of a dark order of size 2 for an informed trader at $t = 1$.

Let θ_t^I and θ_t^U be the probability of execution at t ($t = 1, 2$) for an informed and uninformed trader, respectively. Notice that

$$\theta_1^R = \theta_1^{R,1}, \; R = I, U.$$

In particular, $\theta_1^R \le \frac{1}{2}$, $R = I, U$.

In relation to the probabilities of execution at $t = 2$, we obtain that if there is no change in the book in the LOB (denoted by $\mathcal{B}_1 = \emptyset$), the probability of execution of a BDO for an informed trader is given by:

$$\begin{aligned}\theta_2^I(\mathcal{B}_1 = \emptyset) &= pr\left(\mathcal{B}_1 = BDO|\mathcal{B}_1 = \emptyset\right)\theta_1^{I,2} \\ &\quad + \left(1 - pr\left(\mathcal{B}_1 = BDO|\mathcal{B}_1 = \emptyset\right)\right)\theta_1^{I,1} \\ &= \frac{\lambda\left(\pi\Omega_3 + \frac{(1-\pi)\Gamma_3}{2}\right)}{\lambda\pi\left(\Omega_0 + \Omega_3\right) + \lambda(1-\pi)\left(\Gamma_0 + \Gamma_3\right)}\theta_1^{I,2} \\ &\quad + \left(1 - \frac{\lambda\left(\pi\Omega_3 + \frac{(1-\pi)\Gamma_3}{2}\right)}{\lambda\pi\left(\Omega_0 + \Omega_3\right) + \lambda(1-\pi)\left(\Gamma_0 + \Gamma_3\right)}\right)\theta_1^{I,1},\end{aligned}$$

while if there is a change in the LOB (denoted by $\mathcal{B}_1 \neq \emptyset$), the probability of execution of a BDO for an informed trader satisfies

$$\begin{aligned}\theta_2^I(\mathcal{B}_1 \neq \emptyset) &= pr\left(\mathcal{B}_1 = BDO|\mathcal{B}_1 \neq \emptyset\right)\theta_1^{I,2} \\ &\quad + \left(1 - pr\left(\mathcal{B}_1 = BDO|\mathcal{B}_1 \neq \emptyset\right)\right)\theta_1^{I,1} = \theta_1^{I,1} = \theta_1^I.\end{aligned}$$

In a similar way, we compute the probability of execution of a BDO for an uninformed trader and we obtain:

$$\begin{aligned}\theta_2^U(\mathcal{B}_1 = \emptyset) &= pr\left(\mathcal{B}_1 = BDO|\mathcal{B}_1 = \emptyset\right)\theta_1^{U,2} \\ &\quad + \left(1 - pr\left(\mathcal{B}_1 = BDO|\mathcal{B}_1 = \emptyset\right)\right)\theta_1^{U,1} \\ &= \frac{\lambda\left(\pi\frac{\Omega_3}{2} + \frac{(1-\pi)\Gamma_3}{2}\right)}{\lambda\pi\left(\Omega_0 + \Omega_3\right) + \lambda(1-\pi)\left(\Gamma_0 + \Gamma_3\right)}\theta_1^{U,2} \\ &\quad + \left(1 - \frac{\lambda\left(\pi\frac{\Omega_3}{2} + \frac{(1-\pi)\Gamma_3}{2}\right)}{\lambda\pi\left(\Omega_0 + \Omega_3\right) + \lambda(1-\pi)\left(\Gamma_0 + \Gamma_3\right)}\right)\theta_1^{U,1},\end{aligned}$$

and

$$\theta_2^U(\mathcal{B}_1 \neq \emptyset) = \theta_1^U.$$

Proof of Lemma 2. Note that the set of the possible states of the LOB is the same as in the case there is no DP. However, the state of the book (A_1^1, B_1^1) can be obtained either because a trader arrived and decided not to trade, or because a trader arrived and he submitted a DO.

We solve the model backwards. At $t = 2$ the expected profits of each strategy depend on the state of the LOB. Additionally, uninformed traders form beliefs about the strategies that have been chosen at $t = 1$. Let X and Y be defined as in (B.1) and (B.2), respectively, and Z denote the uninformed trader's belief at $t = 2$ about the probability that a DO that returns to the exchange as a MO at the end of the second trading period was submitted by an informed, which is equal to

$$Z = \frac{(1-\theta_1^I)\pi\Omega_3}{(1-\theta_1^I)\pi\Omega_3 + (1-\theta_1^U)(1-\pi)\Gamma_3}.$$

Table C.1
Expected profits of an informed buyer (IH) at $t = 2$.

IH	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$(\kappa - k_1)\tau$	$\theta_2^I\kappa\tau$	$P_I\delta(k_1 + \kappa - 1)\tau$	0
(A_1^2, B_1^1)	$(\kappa - k_2)\tau$	$\theta_2^I\left(\kappa - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(\kappa - k_1)\tau$	$\theta_2^I\left(\kappa - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$(\kappa - k_1)\tau$	$\theta_2^I\left(\kappa + \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$(\kappa - k_1 + 1)\tau$	$\theta_2^I\left(\kappa + \frac{1}{2}\right)\tau$	0	0

Table C.2
Expected profits of an uninformed buyer (UB) at $t = 2$.

UB	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$-k_1\tau$	0	$P_U\delta(k_1 - Z\kappa - 1)\tau$	0
(A_1^2, B_1^1)	$(X\kappa - k_2)\tau$	$\theta_2^U\left(X\kappa - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(Y\kappa - k_1)\tau$	$\theta_2^U\left(Y\kappa - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$-(X\kappa + k_1)\tau$	$-\theta_2^U\left(X\kappa - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-(Y\kappa + k_1 - 1)\tau$	$-\theta_2^U\left(Y\kappa - \frac{1}{2}\right)\tau$	0	0

As in the case when the DP was not available, and without loss of generality, we will focus on the expected profits for an informed and an uninformed buyer at $t = 2$, as summarized in Table C.1 and Table C.2, respectively.

Define P_I as the probability of execution of a limit order placed by an informed trader at $t = 2$ conditional on the fact that there is no change in the LOB during the first trading period, and equals

$$P_I = p_{BLO,2}^{IH}\left(\mathcal{B}_1 = \emptyset\right) = \frac{(1-\theta_1^U)\frac{1-\pi}{2}\Gamma_3}{\pi\Omega_3 + (1-\pi)(\Gamma_0 + \Gamma_3)}.$$

Define P_U as the probability of execution of a limit order placed by an uninformed trader at $t = 2$ given that there are no changes in prices in the LOB during the first trading period, and equals

$$P_U = p_{BLO,2}^{UB}\left(\mathcal{B}_1 = \emptyset\right) = \frac{1}{2}\frac{(1-\theta_1^I)\pi\Omega_3 + \left(1-\theta_1^U\right)(1-\pi)\Gamma_3}{\pi\Omega_3 + (1-\pi)\left(\Gamma_0 + \Gamma_3\right)}.$$

At $t = 1$ the expected profits of an informed IH and an uninformed buyer UB are summarized in Table C.3 and Table C.4, respectively.[29]

The expressions $I_{SLO,2}^{US,\mathcal{B}_1=\emptyset}$ and $I_{BMO,2}^{IH,\mathcal{B}_1=\emptyset}$ are indicator functions such that $I_{SLO,2}^{US,\mathcal{B}_1=\emptyset} = 1$ if at $t = 2$, an US selects a SLO when the LOB has not changed at $t = 1$, and $I_{SLO,2}^{US,\mathcal{B}_1=\emptyset} = 0$, otherwise. Similarly, the remaining indicator functions can be defined. By simple inspection of the payoffs in Table C.3, it can be seen that an informed buyer at $t = 1$ never chooses NT because it is dominated by a MO.

Notice also that the expected profits of a BDO submitted by an informed buyer at $t = 1$ may be positive, and as a result the informed may choose to place a BDO at $t = 1$ depending on how high the execution probability θ_1^I is. However, the payoff at $t = 1$ of the BDO for the uninformed trader is always null (see Internet Appendix I). Therefore, in the equilibrium path we have that $\Gamma_3 = 0$, and hence $\mathcal{B}_1 = \emptyset$, implies either Ω_3 or Γ_0 is not null. Thus,

$$P_I = p_{BLO,2}^{IH}\left(\mathcal{B}_1 = \emptyset\right) = p_{SLO,2}^{IL}\left(\mathcal{B}_1 = \emptyset\right) = 0.$$

[29] Notice that due to the symmetry of the game, the expected profits of the informed IL trader and uninformed seller US are the same as the ones displayed in Tables C.3 and C.4, respectively.

Table C.3
Expected profits of an informed buyer (IH) at $t=1$.

IH	Expected profits
BMO	$(\kappa-k_1)\,\tau$
BLO	$\frac{\delta(1-\lambda)}{2}(\kappa+k_1-1)\,\tau$
BDO	$\theta_1^I\kappa\tau+(1-\theta_1^I)\delta\left(\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,B_1=\emptyset}+(\kappa-k_1)-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,B_1=\emptyset}+\frac{1-\lambda}{2}\right)\right)\tau$
NT	0

Table C.4
Expected profits of an uninformed buyer (UB) at $t=1$.

UB	Expected profits
BMO	$-k_1\tau$
BLO	$\frac{\delta}{2}\left((1-\lambda)(k_1-1)-\lambda\pi I_{SMO,2}^{IL,B_1=BLO}(\kappa-k_1+1)\right)\tau$
BDO	0
NT	0

Consequently, informed traders never choose a LO at $t=2$, since this order is dominated by a MO. Uninformed traders also do not select a LO at $t=2$. To see this, note that Table C.2 shows that we have to prove the result when prices do not change. In such a case we distinguish two cases: $\Omega_3=1$ and $\Omega_3=0$. In the first case, $Z=1$ and, therefore, the expected profits of a LO are negative, as shown in Table C.2. In the second case, $B_1=\emptyset$ due to $\Gamma_0=1$. Hence,

$$P_U=p_{BLO,2}^{UB}\left(B_1=\emptyset\right)=p_{SLO,2}^{US}\left(B_1=\emptyset\right)=0,$$

and therefore, at $t=2$ the expected profits of a LO for an uninformed trader are null.

Let us determine next the optimal strategy for each rational trader at $t=2$. Depending on the values of the parameters, we have 6 possible cases for the informed trader and 16 cases for the uninformed trader. Due to limits of the length of the paper, the optimal responses of uninformed traders at $t=2$ will be specified in each equilibria (see proof of Lemma C.2), with

$$\theta_X\equiv\frac{X\kappa-k_2}{X\kappa-\frac{k_2-k_1}{2}},\ \text{and}\ \theta_Y\equiv\frac{Y\kappa-k_1}{Y\kappa-\frac{1}{2}}.$$

Next, we focus on informed traders. Given that $\kappa>k_2>k_1\geq 1$, the following inequalities hold:

$$\frac{\kappa-k_2}{\kappa-\frac{k_2-k_1}{2}}<\frac{\kappa-k_1}{\kappa+\frac{k_2-k_1}{2}}<\frac{\kappa-k_1}{\kappa}<\frac{\kappa-k_1}{\kappa-\frac{1}{2}}<\frac{\kappa-k_1+1}{\kappa+\frac{1}{2}}.$$

Hence, the optimal strategies of the informed traders at $t=2$ are given in Table C.5.

We define by

$$BX=\begin{cases}BMO\text{ if }p_{BLO,2}^{IH,B_1=\emptyset}\leq\frac{\kappa-k_1}{\delta(\kappa+k_1-1)},\\ BLO\text{ if }p_{BLO,2}^{IH,B_1=\emptyset}>\frac{\kappa-k_1}{\delta(\kappa+k_1-1)}.\end{cases}$$

$$SX=\begin{cases}SMO\text{ if }p_{SLO,2}^{IL,B_1=\emptyset}\leq\frac{\kappa-k_1}{\delta(\kappa+k_1-1)},\\ SLO\text{ if }p_{SLO,2}^{IL,B_1=\emptyset}>\frac{\kappa-k_1}{\delta(\kappa+k_1-1)}.\end{cases}$$

and

$$BY=\begin{cases}BDO\text{ if }p_{BLO,2}^{IH,B_1=\emptyset}<\frac{\theta_2^I\kappa}{\delta(\kappa+k_1-1)},\\ BLO\text{ if }p_{BLO,2}^{IH,B_1=\emptyset}\geq\frac{\theta_2^I\kappa}{\delta(\kappa+k_1-1)}.\end{cases}$$

$$SY=\begin{cases}SDO\text{ if }p_{BLO,2}^{IH,B_1=\emptyset}<\frac{\theta_2^I\kappa}{\delta(\kappa+k_1-1)},\\ SLO\text{ if }p_{BLO,2}^{IH,B_1=\emptyset}\geq\frac{\theta_2^I\kappa}{\delta(\kappa+k_1-1)}.\end{cases}$$

Table C.5
Optimal strategies of informed traders at $t=2$.

Condition	Optimal Strategies of Informed Traders at t=2		
	State of the book	IH	IL
Case I_1: $\theta_2^I\leq\frac{\kappa-k_2}{\kappa-\frac{k_2-k_1}{2}}$	(A_1^1,B_1^1)	BX	SX
	(A_1^2,B_1^1)	BMO	SMO
	$(A_1^1,B_1^1+\tau)$	BMO	SMO
	(A_1^1,B_1^2)	BMO	SMO
	$(A_1^1-\tau,B_1^1)$	BMO	SMO
Case I_2: $\frac{\kappa-k_2}{\kappa-\frac{k_2-k_1}{2}}<\theta_2^I\leq\frac{\kappa-k_1}{\kappa+\frac{k_2-k_1}{2}}$	(A_1^1,B_1^1)	BX	SX
	(A_1^2,B_1^1)	BDO	SMO
	$(A_1^1,B_1^1+\tau)$	BMO	SMO
	(A_1^1,B_1^2)	BMO	SDO
	$(A_1^1-\tau,B_1^1)$	BMO	SMO
Case I_3: $\frac{\kappa-k_1}{\kappa+\frac{k_2-k_1}{2}}<\theta_2^I\leq\frac{\kappa-k_1}{\kappa}$	(A_1^1,B_1^1)	BX	SX
	(A_1^2,B_1^1)	BDO	SDO
	$(A_1^1,B_1^1+\tau)$	BMO	SMO
	(A_1^1,B_1^2)	BDO	SDO
	$(A_1^1-\tau,B_1^1)$	BMO	SMO
Case I_4: $\frac{\kappa-k_1}{\kappa}<\theta_2^I\leq\frac{\kappa-k_1}{\kappa-\frac{1}{2}}$	(A_1^1,B_1^1)	BY	SY
	(A_1^2,B_1^1)	BDO	SDO
	$(A_1^1,B_1^1+\tau)$	BMO	SMO
	(A_1^1,B_1^2)	BDO	SDO
	$(A_1^1-\tau,B_1^1)$	BMO	SMO
Case I_5: $\frac{\kappa-k_1}{\kappa-\frac{1}{2}}<\theta_2^I\leq\frac{\kappa-k_1+1}{\kappa+\frac{1}{2}}$	(A_1^1,B_1^1)	BY	SY
	(A_1^2,B_1^1)	BDO	SDO
	$(A_1^1,B_1^1+\tau)$	BDO	SMO
	(A_1^1,B_1^2)	BDO	SDO
	$(A_1^1-\tau,B_1^1)$	BMO	SDO
Case I_6: $\frac{\kappa-k_1+1}{\kappa+\frac{1}{2}}<\theta_2^I$	(A_1^1,B_1^1)	BY	SY
	(A_1^2,B_1^1)	BDO	SDO
	$(A_1^1,B_1^1+\tau)$	BDO	SDO
	(A_1^1,B_1^2)	BDO	SDO
	$(A_1^1-\tau,B_1^1)$	BDO	SDO

We next include a definition and a lemma which will be useful to prove Proposition 2.

Definition C.3. Let us consider the following cut-off definitions

$$\hat{\theta}_{MO-DO}(I_{SLO,2}^{US,B_1=\emptyset},I_{BMO,2}^{IH,B_1=\emptyset})\equiv\frac{\kappa-k_1-\delta\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,B_1=\emptyset}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,B_1=\emptyset}+\frac{1-\lambda}{2}\right)\right)}{\kappa-\delta\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,B_1=\emptyset}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,B_1=\emptyset}+\frac{1-\lambda}{2}\right)\right)},$$

$$\hat{\theta}_{LO-DO}(I_{SLO,2}^{US,B_1=\emptyset},I_{BMO,2}^{IH,B_1=\emptyset})\equiv\frac{\delta\frac{1-\lambda}{2}(\kappa+k_1-1)-\delta\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,B_1=\emptyset}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,B_1=\emptyset}+\frac{1-\lambda}{2}\right)\right)}{\kappa-\delta\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,B_1=\emptyset}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,B_1=\emptyset}+\frac{1-\lambda}{2}\right)\right)},$$

$$\underline{\theta}\equiv\frac{\kappa-k_1}{\kappa},\ \text{and}$$

$$\overline{\theta}\equiv\frac{\kappa-k_1+1}{\kappa+\frac{1}{2}},$$

with

$$I_{SLO,2}^{US,B_1=\emptyset}=\begin{cases}1, & \text{if } p_{SLO,2}^{US,B_1=\emptyset}\delta\left(k_1-1-Z\kappa\right)>0,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$I_{BMO,2}^{IH,B_1=\emptyset}=\begin{cases}1, & \text{if } \kappa-k_1\geq\max\left\{\theta_2^I\kappa,p_{BLO,2}^{IH,B_1=\emptyset}\delta\left(k_1+\kappa-1\right)\right\},\\ 0, & \text{otherwise.}\end{cases}$$

Table C.6
Optimal strategies of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Optimal Strategies of Uninformed Traders at t = 2		
State of the Book	UB	US
(A_1^1, B_1^1)	$\begin{cases} NT & \text{if} \quad P = 0 \text{ or } Z^{1,D}\kappa \geq k_1 - 1 \\ BLO & \text{if} \quad P > 0 \text{ and } Z^{1,D}\kappa < k_1 - 1 \end{cases}$	$\begin{cases} NT & \text{if} \quad P = 0 \text{ or } Z^{1,D}\kappa \geq k_1 - 1 \\ SLO & \text{if} \quad P > 0 \text{ and } Z^{1,D}\kappa < k_1 - 1 \end{cases}$
(A_1^2, B_1^1)	$\begin{cases} NT & \text{if} \quad X^{1,D}\kappa \leq \frac{k_2-k_1}{2} \\ BDO & \text{if} \quad \frac{k_2-k_1}{2} < X^{1,D}\kappa \leq k_2 \\ BDO & \text{if} \quad k_2 < X^{1,D}\kappa \text{ and } \theta_2^U > \theta_{X^{1,D}} \\ BMO & \text{if} \quad k_2 < X^{1,D}\kappa \text{ and } \theta_2^U \leq \theta_{X^{1,D}} \end{cases}$	$\begin{cases} SDO & \text{if} \quad X^{1,D}\kappa < \frac{k_2-k_1}{2} \\ NT & \text{if} \quad \frac{k_2-k_1}{2} \leq X^{1,D}\kappa \end{cases}$
$(A_1^1, B_1^1 + \tau)$	NT	SDO
(A_1^1, B_1^2)	$\begin{cases} BDO & \text{if} \quad X^{1,D}\kappa < \frac{k_2-k_1}{2} \\ NT & \text{if} \quad \frac{k_2-k_1}{2} \leq X^{1,D}\kappa \end{cases}$	$\begin{cases} NT & \text{if} \quad X^{1,D}\kappa \leq \frac{k_2-k_1}{2} \\ SDO & \text{if} \quad \frac{k_2-k_1}{2} < X^{1,D}\kappa \leq k_2 \\ SDO & \text{if} \quad k_2 < X^{1,D}\kappa \text{ and } \theta_2^U > \theta_{X^{1,D}} \\ SMO & \text{if} \quad k_2 < X^{1,D}\kappa \text{ and } \theta_2^U \leq \theta_{X^{1,D}} \end{cases}$
$(A_1^1 - \tau, B_1^1)$	BDO	NT

Table C.7
Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Condition	Optimal Choice of Uninformed Traders at t=2		
	State of the book	UB	US
Case $U_1^{\mathcal{E}_1^D}$ $k_1 - 1 \leq \frac{k_2-k_1}{2}$ or $k_1 - 1 > \frac{k_2-k_1}{2}$ and $X^{1,D}\kappa < \frac{k_2-k_1}{2}$	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1 + \tau)$	NT	SDO
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1 - \tau, B_1^1)$	BDO	NT
Case $U_2^{\mathcal{E}_1^D}$ $k_1 - 1 > \frac{k_2-k_1}{2}$ and $X^{1,D}\kappa = \frac{k_2-k_1}{2}$	(A_1^2, B_1^1)	NT	NT
	$(A_1^1, B_1^1 + \tau)$	NT	SDO
	(A_1^1, B_1^2)	NT	NT
	$(A_1^1 - \tau, B_1^1)$	BDO	NT
Case $U_3^{\mathcal{E}_1^D}$ $k_1 - 1 > \frac{k_2-k_1}{2}$ and $\frac{k_2-k_1}{2} < X^{1,D}\kappa < k_1 - 1$	(A_1^2, B_1^1)	BDO	NT
	$(A_1^1, B_1^1 + \tau)$	NT	SDO
	(A_1^1, B_1^2)	NT	SDO
	$(A_1^1 - \tau, B_1^1)$	BDO	NT

Table C.8
Optimal choice of uniformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, NT, NT).

Condition	Optimal Choice of Uninformed Traders at t=2		
	State of the book	UB	US
Case $U_1^{\mathcal{E}_2^D}$ $k_1 - 1 \leq X^{2,D}\kappa < \frac{k_2-k_1}{2}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	SDO
	(A_1^1, B_1^2)	BDO	NT
Case $U_2^{\mathcal{E}_2^D}$ $k_1 - 1 < X^{2,D}\kappa = \frac{k_2-k_1}{2}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	NT
	(A_1^1, B_1^2)	NT	NT
Case $U_3^{\mathcal{E}_2^D}$ $\max\left\{k_1 - 1, \frac{k_2-k_1}{2}\right\} < X^{2,D}\kappa \leq k_2$ or $k_2 < X^{2,D}\kappa$ and $\theta_2^U > \theta_{X^{2,D}}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	BDO	NT
	(A_1^1, B_1^2)	NT	SDO
Case $U_4^{\mathcal{E}_2^D}$ $k_2 < X^{2,D}\kappa$ and $\theta_2^U \leq \theta_{X^{2,D}}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	BMO	NT
	(A_1^1, B_1^2)	NT	SMO

Lemma C.2. ***Case A.*** *Suppose $k_1 > 1$. Then, a PBE of the game is as follows:*

- $\mathcal{E}_1^D$: *(BMO, SMO, BLO, SLO) is the optimal strategy profile at $t = 1$ if*

$$\kappa^I_{MO-LO}\tau \leq \sigma, \; PIN < \psi^U_{LO-NT} \text{ and } \theta_1^I \leq \overline{\theta}^{1,D}, \tag{C.1}$$

with $\overline{\theta}^{1,D} = \widehat{\theta}_{MO-DO}(I^{US,B_1=\emptyset}_{SLO,2}, I^{IH,B_1=\emptyset}_{BMO,2})$. The beliefs of an uninformed trader at $t = 2$ are : $X^{1,D} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{1,D} = 0$ and $Z^{1,D} = z \in [0,1]$. In the equilibrium path, the optimal choice of an uninformed and an informed trader $t = 2$ are described in Table C.7 and a subset of Table C.5, respectively.[30]

- $\mathcal{E}_2^D$: *(BMO, SMO, NT, NT) is the optimal strategy profile at $t = 1$ if*

$$\kappa^I_{MO-LO}\tau \leq \sigma, \; PIN \geq \psi^U_{LO-NT}, \text{ and } \theta_1^I \leq \overline{\theta}^{2,D}, \tag{C.2}$$

with $\overline{\theta}^{2,D} = \widehat{\theta}_{MO-DO}(0,1)$. The beliefs of an uninformed trader at $t = 2$ are : $X^{2,D} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{2,D} = p \in [0,1]$ and $Z^{2,D} = z \in [0,1]$. In the equilibrium path, the optimal choice of an uninformed and an informed trader at $t = 2$ are described in Table C.8 and a subset of Table C.5, respectively.

- $\mathcal{E}_3^D$: *(BLO, SLO, BLO, SLO) is the optimal strategy profile at $t = 1$ if*

$$\begin{aligned} & \sigma < \kappa^I_{MO-LO}\tau, \; PIN < \psi^U_{LO-NT}, \text{ and } \theta_1^I \leq \overline{\theta}^{3,D}, \\ \text{or} \quad & k_1 > 1 \text{ and } \overline{\theta} < \theta_1^I \leq \overline{\theta}^{3,D}, \end{aligned} \tag{C.3}$$

with

$$\overline{\theta}^{3,D} = \begin{cases} \widehat{\theta}_{LO-DO}(I^{US,B_1=\emptyset}_{SLO,2}, I^{IH,B_1=\emptyset}_{BMO,2}) & \text{if} \quad \theta_1^I \leq \underline{\theta}, \\ \widehat{\theta}_{LO-DO}(I^{US,B_1=\emptyset}_{SLO,2}, 0) & \text{if} \quad \theta_1^I > \underline{\theta}. \end{cases}$$

The beliefs of an uninformed trader at $t = 2$ are: $X^{3,D} = 0$, $Y^{3,D} = \pi$ and $Z^{3,D} = z \in [0,1]$. In the equilibrium path, the optimal choice of an uninformed and an informed trader $t = 2$ are described in Table C.9 and a subset of Table C.5, respectively.

- $\mathcal{E}_4^D$: *(BLO, SLO, NT, NT) is the optimal strategy profile of a trader at $t = 1$ if*

$$\sigma < \kappa^I_{MO-LO}\tau, \; PIN \geq \psi^U_{LO-NT}, \text{ and } \theta_1^I \leq \overline{\theta}^{4,D}, \tag{C.4}$$

with

$$\overline{\theta}^{4,D} = \begin{cases} \widehat{\theta}_{LO-DO}(0,1) & \text{if} \quad \theta_1^I \leq \underline{\theta}, \\ \widehat{\theta}_{LO-DO}(0,0) & \text{if} \quad \overline{\theta} \geq \theta_1^I > \underline{\theta}. \end{cases}$$

The beliefs of an uninformed trader at $t = 2$ are: $X^{4,D} = 0$, $Y^{4,D} = 1$ and $Z^{4,D} = z \in [0,1]$. In the equilibrium path, the optimal choice of an uninformed and an informed trader at $t = 2$ are described in Table C.10 and a subset of Table C.5, respectively.

[30] In the proof of the lemma, we describe for each equilibrium the relevant subset of Table C.5.

Table C.9
Optimal choice of uninformed traders when the strategy profile at $t = 1$ is (BLO, SLO, BLO, SLO).

Condition	Optimal Choice of Uninformed Traders at t=2		
	State of the book	UB	US
Case $U_1^{\mathcal{E}_3^D}$ $Y^{3,D}\kappa \le \frac{1}{2}$	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1+\tau)$	NT	SDO
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1-\tau, B_1^1)$	BDO	NT
Case $U_2^{\mathcal{E}_3^D}$ $\frac{1}{2} < Y^{3,D}\kappa \le k_1$ or $Y^{3,D}\kappa > k_1$ and $\theta_2^U > \theta_{Y^{3,D}}$	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1+\tau)$	BDO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1-\tau, B_1^1)$	NT	SDO
Case $U_3^{\mathcal{E}_3^D}$ $Y^{3,D}\kappa > k_1$ and $\theta_2^U \le \theta_{Y^{3,D}}$	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1+\tau)$	BMO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1-\tau, B_1^1)$	NT	SMO

- $\mathcal{E}_5^D$: (BDO, SDO, BLO, SLO) *is the optimal strategy profile of a trader at* $t = 1$ *if*

$$PIN < \psi^U_{LO-NT},\ \overline{\theta} \ge \theta_1^I > \overline{\theta}^{5,D}, \quad \text{or} \quad k_1 > 1,\ \theta_1^I > \max\{\overline{\theta}, \overline{\theta}^{5,D}\}, \tag{C.5}$$

with

$$\overline{\theta}^{5,D} = \begin{cases} \max\{\hat{\theta}_{MO-DO}(0,1), \hat{\theta}_{LO-DO}(0,1)\} & \text{if} \quad \theta_2^I \le \underline{\theta}, \\ \hat{\theta}_{LO-DO}(0,0) & \text{if} \quad \theta_2^I > \underline{\theta}. \end{cases}$$

The beliefs of an uninformed trader at $t = 2$ *are* : $X^{5,D} = 0$, $Y^{5,D} = 0$ *and* $Z^{5,D} = 1$. *In the equilibrium path, the optimal choice of an uninformed and an informed trader* $t = 2$ *are described in* Table C.11 *and a subset of* Table C.5, *respectively.*
- $\mathcal{E}_6^D$: (BDO, SDO, NT, NT) *is the optimal strategy profile of a trader at* $t = 1$ *if*

$$PIN \ge \psi^U_{LO-NT} \text{ and } \overline{\theta} \ge \theta_1^I > \overline{\theta}^{6,D}, \tag{C.6}$$

with $\overline{\theta}^{6,D} = \overline{\theta}^{5,D}$. *The beliefs of an uninformed trader at* $t = 2$ *are:* $X^{6,D} = 0$, $Y^{6,D} = p \in [0,1]$ *and* $Z^{6,D} = 1$. *In the equilibrium path, the optimal choice of an uninformed and an informed trader at* $t = 2$ *are described in* Table C.12 *and a subset of* Table C.5, *respectively.*

Case B. *Suppose* $k_1 = 1$. *Then,* (BMO, SMO, NT, NT) *is the optimal strategy profile at* $t = 1$ *if* $\theta_1^I \le \overline{\theta}^{2,D}$, *and* (BDO, SDO, NT, NT) *is the optimal strategy profile at* $t = 1$ *if* $\theta_1^I > \overline{\theta}^{6,D}$.

Remark C.1. Recall that in a Perfect Bayesian Equilibrium beliefs must satisfy Bayes' rule, whenever possible. This occurs along the equilibrium path, not off-the-equilibrium path, where beliefs are indeterminate. This indeterminacy might result in multiplicity of equilibria in sequential games with imperfect information. Note that this may occur in our case when the uninformed trader's beliefs at $t = 2$ (i.e., X, Y, or Z) are indeterminate.

Proof of Lemma C.2. Because of the symmetry of the model, without any loss of generality, we focus on buyers. We present the full proof for one of the possible strategy profile at $t = 1$ that yields an equilibrium. The proofs of all the other 5 equilibria can be found in the Internet Appendix II. Note that in all equilibria the optimal responses of informed traders at $t = 2$ are given in Table C.5.

However, in some equilibria not all the 6 cases $I_1 - I_6$ are possible and also not all of the 5 states of the book are possible. As a result only a subset of Table C.5 will apply.

$\mathcal{E}_1^D$: (BMO, SMO, BLO, SLO)

First step. In this case $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$, $\Omega_3 = 0$, $\Gamma_0 = 0$, $\Gamma_1 = 0$, $\Gamma_2 = 1$, and $\Gamma_3 = 0$. Moreover, $\theta_2^I = \theta_1^I$ and $\theta_2^U = \theta_1^U$. We define by

Table C.10
Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

Condition	Optimal Choice of Uninformed Traders at t=2		
	State of the book	UB	US
Case $U_1^{\mathcal{E}_4^D}$ $\theta_2^U > \theta_{Y^{4,D}}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1+\tau)$	BMO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1-\tau, B_1^1)$	NT	SMO
Case $U_2^{\mathcal{E}_4^D}$ $\theta_2^U \le \theta_{Y^{4,D}}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1+\tau)$	BDO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1-\tau, B_1^1)$	NT	SDO

Table C.11
Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, BLO, SLO).

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
$(A_1^1, B_1^1+\tau)$	NT	SDO
(A_1^1, B_1^2)	BDO	NT
$(A_1^1-\tau, B_1^1)$	BDO	NT

Table C.12
Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, NT, NT).

Optimal Choice of Uninformed Traders at $t = 2$		
State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
(A_1^1, B_1^2)	BDO	NT

$P \equiv p^{UB,\mathcal{B}_1=\emptyset}_{BLO,2} = p^{US,\mathcal{B}_1=\emptyset}_{SLO,2}$.

Second step. Using Bayes' rule,

$$X^{1,D} = \frac{\lambda\pi}{1-\lambda+\lambda\pi},\ Y^{1,D} = 0,\ Z^{1,D} = z \in [0,1],$$
$$p^{UB,\mathcal{B}_1=\emptyset}_{BLO,2} = p^{US,\mathcal{B}_1=\emptyset}_{SLO,2} \in [0,1], \text{ and } p^{IH,\mathcal{B}_1=\emptyset}_{BLO,2} = p^{IL,\mathcal{B}_1=\emptyset}_{SLO,2} \in [0,1].$$

Third step. Using step 2 and taking into account that $p^{UB}_{BLO,2}(\mathcal{B}_1 = \emptyset) = p^{US}_{SLO,2}(\mathcal{B}_1 = \emptyset) \in [0,1]$, at $t = 2$ the expected profits of uninformed buyers are as given by Table C.2. Using the symmetry of buyers and sellers, Table C.6 gives us the optimal strategy for the uninformed.

Concerning the informed buyers their expected profits are as given by Table C.1 and the optimal strategy for an informed trader at $t = 2$ is given by Table C.5.

Fourth step. Given the optimal response of traders at $t = 2$, we find the optimal action for the traders at $t = 1$ in each of the 6 cases. However, given the nature of this particular equilibrium, we can group cases and analyze them in the following way:

Case $I_1 + I_2 + I_3$: $\theta_2^I \le \frac{\kappa - k_1}{\kappa}$

- *Informed traders*
  As $\theta_2^I \le \frac{\kappa - k_1}{\kappa}$, informed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$\kappa - k_1 \ge \frac{1-\lambda}{2}\delta\left(\kappa + k_1 - 1\right) \text{ and} \tag{C.7}$$
$$\kappa - k_1 \ge \theta_1^I\kappa + (1-\theta_1^I)\delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I^{US,\mathcal{B}_1=\emptyset}_{SLO,2} - (k_2 - k_1)\left(\lambda\pi I^{IH,\mathcal{B}_1=\emptyset}_{BMO,2} + \frac{1-\lambda}{2}\right)\right).$$

- *Uninformed traders*
As $\theta_2^I \leq \frac{\kappa - k_1}{\kappa} \leq \frac{\kappa - k_1 + 1}{\kappa + \frac{1}{2}}$, uninformed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$(\lambda\pi + 1 - \lambda)\left(k_1 - 1\right) - \lambda\pi\kappa > 0. \tag{C.8}$$

Case $I_4 + I_5 + I_6$: $\frac{\kappa - k_1}{\kappa} < \theta_2^I$

- *Informed traders*

Consider an informed buyer at $t = 1$. If he chooses a BMO, then he obtains

$$\mathbb{E}\left(\Pi^{IH}_{BMO,1}\right) = \left(\kappa - k_1\right)\tau.$$

If instead he deviates towards a BDO, he will obtain

$$\mathbb{E}\left(\Pi^{IH}_{BDO,1}\right) = \theta_1^I\kappa\tau + (1-\theta_1^I)\delta\left[\lambda\frac{(1-\pi)}{2}I^{US,B_1=\varnothing}_{SLO,2} + \left(\kappa - k_1\right) - \left(k_2 - k_1\right)\left(\lambda\pi I^{IH,B_1=\varnothing}_{BMO,2} + \frac{1-\lambda}{2}\right)\right]\tau.$$

Combining the previous expression and the fact that $\frac{\kappa - k_1}{\kappa} < \theta_2^I = \theta_1^I$, it follows that

$$\mathbb{E}\left(\Pi^{IH}_{BDO,1}\right) > \mathbb{E}\left(\Pi^{IH}_{BMO,1}\right) \tag{C.9}$$

is always satisfied, and we conclude that in this case there is no equilibrium in which the strategy profile chosen at $t = 1$ is (BMO, SMO, BLO, SLO).

Fifth step. Based on the above, nobody at $t = 1$ has unilateral incentives to deviate whenever $\theta_1^I \leq \frac{\kappa - k_1}{\kappa}$ and (C.7), (C.8) and (C.9) are satisfied. These conditions can be rewritten as the ones in (C.1).

Finally, we include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BMO, SMO, BLO, SLO) is the strategy profile chosen at $t = 1$ and the fact that in this case $\theta_2^I = \theta_1^I$. In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 the cases I_1, I_2 and I_3 and the following possible prices: (A_1^2, B_1^1), $(A_1^1, B_1^1+\tau)$, (A_1^1, B_1^2), $(A_1^1-\tau, B_1^1)$.

Concerning uninformed traders notice that the condition $(\lambda\pi + 1 - \lambda)\left(k_1 - 1\right) - \lambda\pi\kappa > 0$ implies that $X^{1,D}\kappa < k_1 - 1 < k_2$. Hence, the optimal choice of uninformed traders at $t = 2$ are in Table C.7.

$\mathcal{E}_2^D$: (BMO, SMO, NT, NT)

In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{2,D} = \frac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{2,D} = p \in [0,1]$ and $Z^{2,D} = z \in [0,1]$. In addition, nobody at $t = 1$ has unilateral incentives to deviate whenever the conditions in (C.2) are satisfied.

Finally, we include the decisions that are in the equilibrium path and that in this case $\theta_2^I = \theta_1^I$. In relation to uninformed traders, and taking into account that $k_1 - 1 \leq X^{2,D}\kappa$, we obtain Table C.8.

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table C.5 only the cases I_1, I_2 and I_3, and the following possible prices: (A_1^1, B_1^1), (A_1^2, B_1^1), and (A_1^1, B_1^2), with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

$\mathcal{E}_3^D$: (BLO, SLO, BLO, SLO)

In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{3,D} = 0$, $Y^{3,D} = \pi$, and $Z^{3,D} = z \in [0,1]$. In addition, the conditions under which nobody is willing to deviate at $t = 1$ are given in (C.3).

Finally, we include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BLO, SLO, BLO, SLO) is the strategy profile chosen at $t = 1$ and $\theta_2^I = \theta_1^I$. Concerning uninformed traders, it follows that their optimal choices at $t = 2$ are given in Table C.9.

In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 all the cases I_1– I_6 and the prices: (A_1^2, B_1^1), $(A_1^1, B_1^1 + \tau)$, (A_1^1, B_1^2), and $(A_1^1 - \tau, B_1^1)$.

$\mathcal{E}_4^D$: (BLO, SLO, NT, NT)

The beliefs of an uninformed trader at $t = 2$ are: $X^{4,D} = 0$, $Y^{4,D} = 1$, and $Z^{4,D} = z \in [0,1]$. In addition, the conditions under which nobody is willing to deviate at $t = 1$ are given in (C.4).

We include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BLO, SLO, NT, NT) is the strategy profile chosen at $t = 1$ and the fact that in this case $\theta_2^I = \theta_1^I$ and $\theta_2^U = \theta_1^U$. Concerning the uninformed traders at $t = 2$, we have the optimal choices in Table C.10.

In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 all the cases I_1– I_5, for all the possible pairs of best prices, with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

$\mathcal{E}_5^D$: (BDO, SDO, BLO, SLO)

In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{5,D} = 0$, $Y^{5,D} = 0$, and $Z^{5,D} = 1$. In addition, nobody at $t = 1$ has unilateral incentives to deviate from (BDO, SDO, BLO, SLO) whenever the conditions in (C.5) are satisfied.

Notice that in this equilibrium we always have $\theta_2^I \leq \theta_1^I$. Furthermore, the optimal responses of uninformed traders are in Table C.11.

In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 all the cases I_1– I_6, for all the possible pairs of best prices, with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

$\mathcal{E}_6^D$: (BDO, SDO, NT, NT)

In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{6,D} = 0$, $Y^{6,D} = p \in [0,1]$ and $Z^{6,D} = 1$. In addition, nobody at $t = 1$ has unilateral incentives to deviate from (BDO, SDO, NT, NT) whenever either the conditions in (C.6) or $k_1 = 1$ and $\theta_1^I > \max\{\overline{\theta}, \overline{\theta}^{6,D}\}$ are satisfied.

Notice that in this equilibrium we also have that $\theta_2^I \leq \theta_1^I$. Furthermore, the optimal responses of uninformed traders are in Table C.12.

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table C.5 all the cases $I_1 - I_6$ and the following possible prices: (A_1^1, B_1^1), (A_1^2, B_1^1), (A_1^1, B_1^2), with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

Case B. Note that substituting $k_1 = 1$ into the expressions of κ^I_{MO-LO} and ψ^U_{LO-NT}, we have that

$$\kappa^I_{MO-LO} = \frac{1}{1 - \frac{1}{2}\delta(1-\lambda)} \text{ and } \psi^U_{LO-NT} = 0.$$

Moreover, since $\kappa^I_{MO-LO} < 2$, it follows that $\kappa^I_{MO-LO}\tau < \sigma$ and $PIN \geq \psi^U_{LO-NT}$. Therefore, using (C.1)–(C.6), we have that when $k_1 = 1$, the conditions related to $\mathcal{E}_1^D$, $\mathcal{E}_3^D$ and $\mathcal{E}_5^D$ do not hold. Moreover, when $k_1 = 1$, $\delta\frac{1-\lambda}{2}\kappa < \kappa - 1$, which implies that an informed trader at $t = 1$ prefers a MO to a LO. Hence, $\mathcal{E}_4^D$ is not feasible when $k_1 = 1$. Therefore, in this case we have that (BMO, SMO, NT, NT) is the optimal strategy profile at $t = 1$ if $\theta_1^I \leq \overline{\theta}^{2,D}$; and (BDO, SDO, NT, NT) is the optimal strategy profile at $t = 1$ if $\theta_1^I > \overline{\theta}^{6,D}$.

Proof of Proposition 2. We consider the same four possible cases depending on the initial conditions in the single-venue market.

Case A.1: $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$

Proposition 1 shows that in the single-venue market the equilibrium is $\mathcal{E}_3^{ND}$. In this case, when we add the DP out of the 6 equilibria, there are only two possible equilibria that satisfy these conditions: $\mathcal{E}_3^D$ and $\mathcal{E}_5^D$. From Lemma C.2 we can see that these equilibria arise if conditions (C.3) and (C.5) are satisfied, respectively. Therefore, when $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$, the optimal strategy profiles at $t = 1$ are

$$\begin{cases} (BLO, SLO, BLO, SLO), & \text{if} \quad \theta_1^I \leq \overline{\theta}^{3,D}, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \overline{\theta}^{5,D}. \end{cases}$$

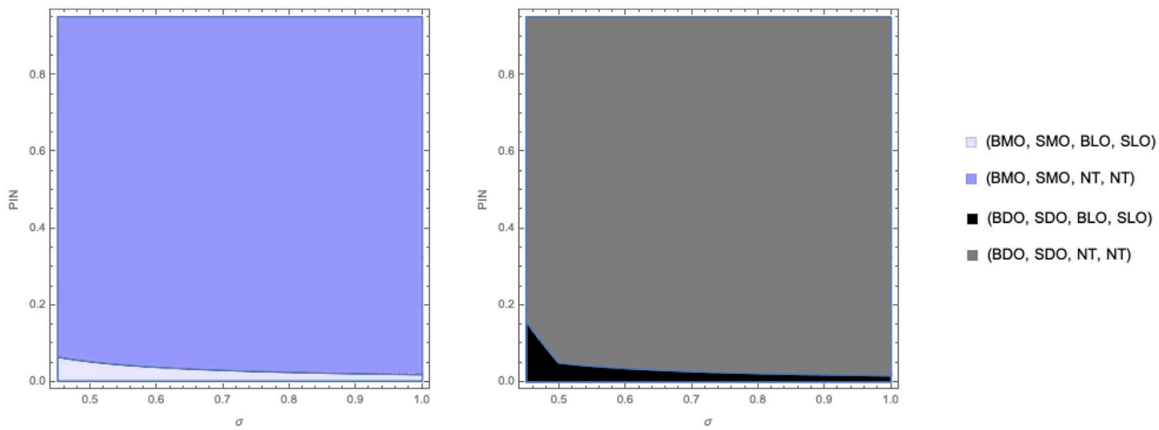

Fig. D.1. Optimal strategies at $t = 1$ with dark pool. Parameters values: $k_1 = 6$, $k_2 = 7$, $\lambda = 0.95$, $\tau = 0.05$, $\delta = 0.95$. In the left panel $\theta_1^I = 0$, in the right panel $\theta_1^I = 0.5$.

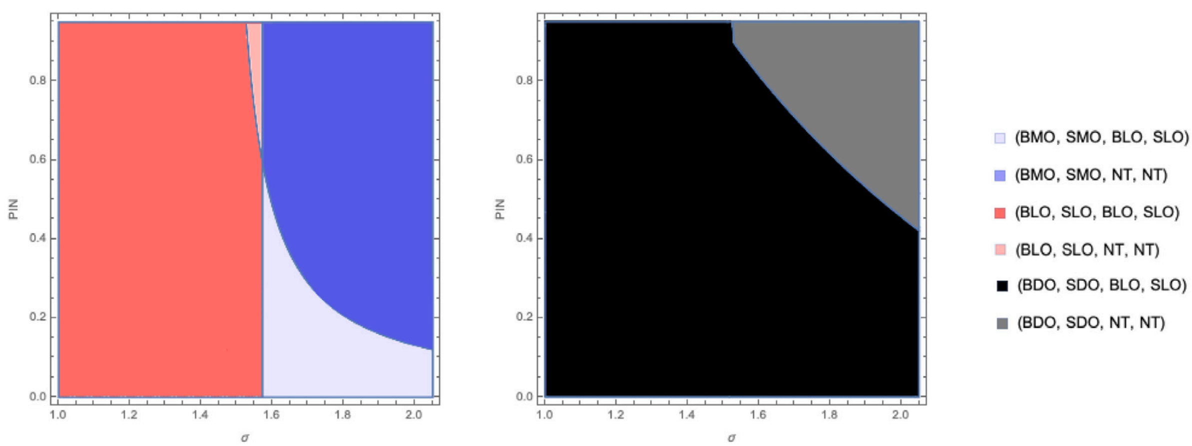

Fig. D.2. Optimal strategies at $t = 1$ with dark pool. Parameters values: $k_1 = 30$, $k_2 = 31$, $\lambda = 0.9$, $\tau = 0.05$, $\delta = 0.95$. In the left panel $\theta_1^I = 0$, in the right panel $\theta_1^I = 0.5$.

Case A.2: $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \geq \psi^U_{LO-NT}$

Proposition 1 shows that in the single-venue market the equilibrium is $\mathcal{E}_4^{ND}$. In this case, when we add the DP out of the 6 equilibria there are only three possible equilibria that satisfy these conditions: $\mathcal{E}_4^D$, $\mathcal{E}_5^D$, and $\mathcal{E}_6^D$. From Lemma C.2 we can see that these equilibria arise if conditions (C.4), (C.5) and (C.6) are satisfied, respectively. Therefore, when $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \geq \psi^U_{LO-NT}$, the optimal strategy profiles at $t = 1$ are

$$\begin{cases} (BLO, SLO, NT, NT) & \text{if} \quad \theta_1^I \leq \overline{\theta}^{4,D}, \\ (BDO, SDO, NT, NT) & \text{if} \quad \overline{\theta} \geq \theta_1^I > \overline{\theta}^{5,D}, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \max\{\overline{\theta}, \overline{\theta}^{5,D}\}. \end{cases}$$

Case A.3: $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN < \psi^U_{LO-NT}$

Proposition 1 shows that in the single-venue market the equilibrium is $\mathcal{E}_1^{ND}$. In this case, when we add the DP out of the 6 equilibria, there are only two possible equilibria that satisfy these conditions: $\mathcal{E}_1^D$ and $\mathcal{E}_5^D$. From Lemma C.2 we can see that these equilibria arise if conditions (C.1) and (C.5) are satisfied, respectively. Therefore, when $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN < \psi^U_{LO-NT}$, the optimal strategy profiles at $t = 1$ are

$$\begin{cases} (BMO, SMO, BLO, BLO) & \text{if} \quad \theta_1^I \leq \overline{\theta}^{1,D}, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \overline{\theta}^{5,D}. \end{cases}$$

Case A.4: $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$

Proposition 1 shows that in the single-venue market the equilibrium is $\mathcal{E}_2^{ND}$. In this case, when we add the DP out of the 6 equilibria, there are only two possible equilibria that satisfy these conditions: $\mathcal{E}_2^D$, $\mathcal{E}_5^D$, and $\mathcal{E}_6^D$. From Lemma C.2 we can see that these equilibria arise if conditions (C.2), (C.5) and (C.6) are satisfied, respectively. Therefore, when $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$, the optimal strategy profiles at $t = 1$ are

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta_1^I \leq \overline{\theta}^{2,D}, \\ (BDO, SDO, NT, NT) & \text{if} \quad \overline{\theta} \geq \theta_1^I > \overline{\theta}^{5,D}, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \max\{\overline{\theta}, \overline{\theta}^{5,D}\}. \end{cases}$$

Case B. From Lemma C.2, we know that in this case there are only two possible equilibria when there is access to the DP: $\mathcal{E}_2^D$ and $\mathcal{E}_6^D$. In addition, when $k_1 = 1$ the optimal strategy profiles at $t = 1$ are

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta_1^I \leq \overline{\theta}^{2,D}, \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta_1^I > \overline{\theta}^{6,D}. \end{cases}$$

Proofs of Propositions 3–8. See Internet Appendix III.

Appendix D. Additional graphs

In this Appendix, we present additional graphs which complement Fig. 4. In Fig. D.1 and Fig. D.2 we focus on the region where an uninformed trader switches from NT to a LO.

Appendix E. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.econmod.2023.106376.

References

Anand, A., Samadi, M., Sokobin, J., Venkataraman, K., 2021. Institutional order handling and broker-affiliated trading venues. Rev. Financ. Stud. 34, 3364–3402.

Bank of America Securities, 2021. US equities market execution venues. October. Retrieved 27 January, 2023, from https://business.bofa.com/content/dam/flagship/global-markets/high-touch-electronic-trading/documents/us_equity_execution_venues.pdf.

Battalio, R., Corwin, S.A., Jennings, R., 2016. Can brokers have it all? On the relation between make-take fees and limit order execution quality. J. Finance 5, 2193–2238.

Biais, B., 1993. Price formation and equilibrium liquidity in fragmented and centralized markets. J. Finance 48, 157–185.

big xyt, 2022. European equities market microstructure survey. Retrieved 27 January, 2023, from https://xytview.com.

Boulatov, A., George, T.J., 2013. Hidden and displayed liquidity in securities markets with informed liquidity providers. Rev. Financ. Stud. 26, 2096–2137.

Brogaard, J., Pan, J., 2021. Dark trading and information acquisition. Rev. Financ. Stud. 35, 2625–2666.

Brolley, M., 2020. Price improvement and execution risk in lit and dark markets. Manage. Sci. 66, 863–886.

Brolley, M., Malinova, K., 2021. Informed liquidity provision in a limit order market. J. Financ. Mark. 52, 100566.

Buti, S., Rindi, B., 2013. Undisclosed orders and optimal submission strategies in a limit order market. J. Financ. Econ. 109, 797–812.

Buti, S., Rindi, B., Werner, I.M., 2017. Dark pool trading strategies, market quality and welfare. J. Financ. Econ. 124, 244–265.

Buti, S., Rindi, B., Werner, I.M., 2022. Diving into dark pools. Financ. Manage 51, 961–994.

Chakravarty, S., Holden, C., 1995. An integrated model of market and limit orders. J. Financial Inter. 4, 213–241.

Cohen, L., Frazzini, A., Malloy, C., 2008. The small world of investing: Board connections and mutual fund returns. J. Polit. Econ. 116, 951–979.

Comerton-Forde, C., Malinova, K., Park, A., 2018. Regulating dark trading: Order flow segmentation and market quality. J. Financ. Econ. 130, 347–366.

Comerton-Forde, C., Putniņš, T., 2015. Dark trading and price discovery. J. Financ. Econ. 118, 70–92.

Coval, J., Moskowitz, T., 2001. The geography of investment: Informed trading and asset prices. J. Polit. Econ. 4, 811–841.

Degryse, H., De Jong, F., Van Kervel, V., 2015. The impact of dark trading and visible fragmentation on market quality. Rev. Finance 19, 1587–1622.

Degryse, H., Van Achter, M., Wuyts, G., 2009. Dynamic order submission strategies with competition between a dealer market and a crossing network. J. Financ. Econ. 91, 319–338.

Dumitrescu, A., 2010. Liquidity and optimal market transparency. Eur. Financial Manag. 16, 599–623.

Easley, D., Kiefer, N.M., O'Hara, M., Paperman, J.B., 1996. Liquidity, information, and infrequently traded stocks. J. Finance 51, 1405–1436.

Easley, D., O'Hara, M., 1987. Price, trade size, and information in securities markets. J. Financ. Econ. 19, 69–90.

Foley, S., Putniņš, T., 2016. Should we be afraid of the dark? Dark trading and market quality. J. Financ. Econ. 122, 456–481.

Foucault, T., 1999. Order flow composition and trading costs in a dynamic limit order market. J. Financial Mark. 2, 99–134.

Foucault, T., Kadan, O., Kandel, E., 2005. Limit order book as a market for liquidity. Rev. Financ. Stud. 18, 1171–1217.

Frutos, M.A., Manzano, C., 2002. Risk aversion, transparency, and market performance. J. Finance 57, 959–984.

Frutos, M.A., Manzano, C., 2005. Trade disclosure and price experimentation. J. Financial Mark. 8, 183–216.

Garvey, R., Huang, T., Wu, F., 2016. Why do traders choose dark markets? J. Bank. Financ. 68, 12–28.

Glosten, L.R., 1994. Is the electronic open limit order book inevitable? J. Finance 49, 1127–1161.

Glosten, L.R., Milgrom, P.R., 1985. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. J. Financ. Econ. 14, 71–100.

Goettler, R.L., Parlour, C.A., Rajan, U., 2009. Informed traders and limit order markets. J. Financ. Econ. 93, 67–87.

Goldman Sachs, 2022. Sigma X^2 monthly metrics. Retrieved 27 January, 2023, from https://www.goldmansachs.com/what-we-do/FICC-and-equities/gset/equities/liquidity-access/sigma-x-us-monthly.pdf.

Gomber, P., Sagade, S., Theissen, E., Weber, M.C., Westheide, C., 2016. Competition between equity markets: A review of the consolidation versus fragmentation debate. J. Econ. Surv. 31, 792–814.

Gresse, C., 2006. The effect of crossing-network trading on dealer market's bid–ask spreads. Eur. Financial Manag. 12, 143–160.

Gresse, C., 2017. Effects of lit and dark market fragmentation on liquidity. J. Financial Mark. 35, 1–20.

Han, B., Tang, Y., Yang, L., 2016. Public information and uninformed trading: Implications for market liquidity and price efficiency. J. Econ. Theory 163, 604–643.

Hatheway, F., Kwan, A., Zhen, H., 2017. An empirical analysis of market segmentation on U.S. equities markets. J. Financial Quant. Anal. 52, 2399–2427.

He, W.P., Lepone, A., 2014. Determinants of liquidity and execution probability in exchange operated dark pool: Evidence from the Australian Securities Exchange. Pacific Basin Finance J. 30, 1–16.

Hendershott, T., Jones, C.M., 2005. Island goes dark: Transparency, fragmentation, and regulation. Rev. Financ. Stud. 18, 743–793.

Hendershott, T., Mendelson, H., 2000. Crossing networks and dealer markets: Competition and performance. J. Finance 55, 2071–2115.

Hendershott, T., Wee, M., Wen, Y., 2022. Transparency in fragmented markets: Experimental evidence. J. Financial Mark. 59, 100732.

J.P. Morgan, 2022. JPM-X U.S. Quarterly summary Q3 2022. Retrieved 27 January, 2023, from https://www.jpmorgan.com/solutions/cib/markets/jpm-x-jpb-x-us-quarterly-summary.

Kaniel, R., Liu, H., 2006. So what orders do informed traders use? J. Bus. 79, 1867–1914.

Kwan, A., Masulis, R., McInish, T.H., 2015. Trading rules, competition for order flow and market fragmentation. J. Financ. Econ. 115, 330–348.

Level ATS, 2022. Monthly aggregate trading data. December. Retrieved 27 January, 2023, from https://www.kezarmarkets.com/december-2022-monthly-aggregate-trading-data/s.

Madhavan, A., 1995. Consolidation, fragmentation, and the disclosure of trading information. Rev. Financ. Stud. 8, 579–603.

Menkveld, A., Yueshen, B., Zhu, H., 2017. Shades of darkness: A pecking order of trading venues. J. Financ. Econ. 124, 503–534.

Naes, R., Odegaard, B.A., 2006. Equity trading by institutional investors: To cross or not to cross? J. Financial Mark. 9, 79–99.

Nimalendran, M., Ray, S., 2014. Informational linkages between dark and lit trading venues. J. Financial Mark. 17, 230–261.

Parlour, C.A., 1998. Price dynamics in limit order markets. Rev. Financ. Stud. 11, 789–816.

Parlour, C.A., Seppi, D.J., 2003. Liquidity-based competition for order flow. Rev. Financ. Stud. 16, 301–343.

Petrescu, M., Wedow, M., 2017. Dark pools in European equity markets: Emergence, competition and implications. ECB Occasional Paper 193.

Ready, M., 2014. Determinants of volume in dark pool crossing networks. AFA 2010 Atlanta Meetings Paper, Available at SSRN: https://ssrn.com/abstract=1361234.

Riccó, R., Rindi, B., Seppi, D.J., 2020. Information, liquidity, and dynamic limit order markets. Available at SSRN: https://ssrn.com/abstract=3032074.

Rosenblatt Securities, 2023. Let there be light. In: Market Structure Analysis. Retrieved 27 January, 2023, from https://www.rblt.com/market-reports/let-there-be-light-us-edition-44.

Rosu, I., 2009. A dynamic model of the limit order book. Rev. Financ. Stud. 22, 4601–4641.

Seppi, D.J., 1997. Liquidity provision with limit orders and a strategic specialist. Rev. Financ. Stud. 10, 103–150.

UBS, 2022. UBS ATS monthly volume summary. Retrieved 27 January, 2023, from https://www.ubs.com/global/en/investment-bank/electronic-trading/archive.html.

Virtu Financial, 2022a. APAC market structure monthly. Retrieved 27 January, 2023, from https://www.virtu.com.

Virtu Financial, 2022b. Canada market structure monthly. Retrieved 27 January, 2023, from https://www.virtu.com.

Virtu Financial, 2022c. U.S. market structure monthly. Retrieved 27 January, 2023, from https://www.virtu.com.

Weaver, D.G., 2014. The trade-at rule, internalization, and market quality. Available at SSRN: https://ssrn.com/abstract=1846470.

Ye, M., 2011. A glimpse into the dark: Price formation, transaction cost and market share of the crossing network. Available at SSRN: https://ssrn.com/abstract=1521494.

Ye, M., Zhu, W., 2020. Strategic informed trading and dark pools. Available at SSRN: https://ssrn.com/abstract=3292516.

Zhu, H., 2014. Do dark pools harm price discovery? Rev. Financ. Stud. 27, 747–789.