Viscoelastic models revisited: characteristics and interconversion formulas for generalized Kelvin-Voigt and Maxwell models

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Abstract: Generalized Kelvin-Voigt and Maxwell models using Prony series are one of the most known models to characterize the behavior of polymers. The simulation software for viscoelastic materials generally implement only some material models. Therefore, for the practice of the engineer, it is very useful to have formulas that establish the equivalence between different models. Although the existence of these relationships is a well-established fact, moving from one model to another involves a relatively long process. This article presents a development of the relationships between generalized Kelvin-Voigt and Maxwell models using the aforementioned series, and their respective relaxation and creep coefficients for one and two summations. The relationship between the singular points (maximums, minimums and inflexion points) is also included.

Keywords: viscoelasticity, Dynamic Mechanical Analysis (DMA), mechanical vibrations.

Introduction

The mathematical models to characterize mechanical properties of the materials are a permanent issue in engineering research. The simplest and more friendly models for this proposal are the Kelvin-Voigt and Maxwell models [1–5], because their linearity. Both models are not equivalent: the first one can explain creep but not relaxation phenomena, and the second one does the opposite. Both models correspond to materials that show only one characteristic time. In order to fit a model to materials showing several characteristic times, generalized linear viscoelastic models are used, connecting several Kelvin-Voigt or Maxwell elements. Every generalized Maxwell (GM) model has an equivalent generalized Kelvin-Voigt (GKV) model. These models correspond to the use of Prony series to adjust creep and relaxation functions. This is a usual option in a variety of engineering applications involving viscoelastic materials [6–16], and the identification of the parameters of the material models is nowadays still a challenging issue [17, 18, 27, 28, 19–26].

The practitioner engineer, when using different tools for calculation, must use these viscoelastic models under a specific form [29]. This fact can be relatively important, because commercial software is usually implemented with only some specific viscoelastic models. Therefore, it is necessary for the user to know the conversions procedure to pass from one model to another. Although the existence of these relationships is a well-known fact [4, 5, 30–32], it usually implies some numerical methods. A number of these numerical methods have been developed and published [32–37]. A recent work [37] offers an interconversion method between the different models by explicit expressions, except for the determination of the zeros of certain polynomials. For some cases it is possible to establish quite explicit formulas of interconversion, which are presented in the consulted literature [4] but only for the simplest cases.

This work presents the complete set of explicit expressions of the relationships between GKV and GM models of order one and two, as well as relaxation and creep coefficients. To the best of our knowledge, there is not a set of explicit formulas published as we propose. From this study, it is also possible to study singular points as maximums, minimums or inflexion points.

These relationships allow to move from one model to another equivalent easily and quickly.

1. Characterization of Prony series

In this section, the generalized Kelvin-Voigt and Maxwell models as combination of springs and dashpots are characterized.

In the GKV model, only creep, storage and loss compliances are presented. The other way round occurs in the GM model, whose values presented are relaxation, storage and loss moduli. As the number of terms n in the models is increased, the equations found in the Laplace domain are more complicated to be expressed in the time or frequency domains. Only in two cases (in GKV model with compliances and in GM model with moduli), these transformations are simple, since every term in the Laplace domain can be separately transformed in an inverse exponential in the time domain.

The difficulty arises to pass from the coefficients of the creep function to the relaxation moduli, for example. Conversely, the same happens when it is interesting to obtain the compliance coefficients from the relaxation function. In this work, these interconversions are solved when $n = 1$ or $n = 2$, in an explicit form. However, in other cases, with $n > 2$, the expressions as a function of n in the Laplace domain contain a denominator with a polynomial equation of degree n. There is no expression to pass easily from Laplace

domain to time domain when a polynomial equation of degree three or more appears in the denominator. For this reason, all coefficients are just solved for $n = 1$ and $n = 2$.

In the following sub-sections, the explicit expressions to obtain the coefficients of the relaxation function from the KVG model for n = 1 and n = 2 have been developed. Also, explicit expressions for the creep coefficients from the GM model up to second order have been obtained. Moreover, the general expressions of compliances for GKV models and moduli for GM models are shown.

1.1 Generalized Kelvin-Voigt model

A Kelvin-Voigt element is a set of one spring and one dashpot connected in parallel. The generalized Kelvin-Voigt model is a series of Kelvin-Voigt elements with a spring, all them connected in series (figure 1).

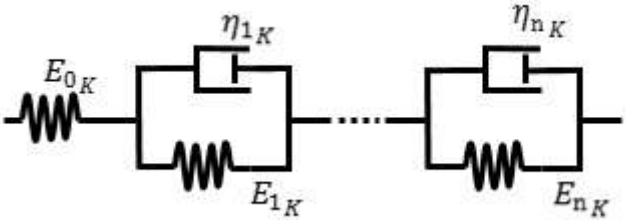

Figure 1 - Generalized Kelvin-Voigt model of order n

The elastic (E) and viscous (η) parameters are defined as usual:

$$\sigma = E \cdot \varepsilon \tag{1a}$$

$$\sigma = \eta \cdot \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{1b}$$

where σ is the stress and ε the strain, and the prime indicate the time derivative. The stress along the model is the same for each block in figure 1, and the total strain of the model results of the strain summation of every block.

1.1.1 First order generalized Kelvin-Voigt model

By definition, the GKV model of first order has the representation of figure 2. This is a material model commonly used, and is also known as linear standard model, or three-parameter model.

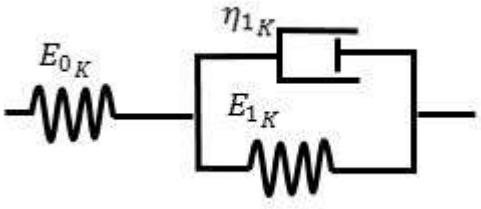

Figure 2 – Generalized Kelvin-Voigt model with $n = 1$

Considering the expressions (1), the Laplace transform leads to:

$$E_{0_K} \cdot \mathcal{L}[\varepsilon_0] = \left(E_{1_K} + \eta_{1_K} \cdot s\right) \cdot \mathcal{L}[\varepsilon_1] \tag{2a}$$

$$\mathcal{L}[\varepsilon] = \mathcal{L}[\varepsilon_0] + \frac{E_{0K}}{E_{1_K}+\eta_{1_K} \cdot s} \cdot \mathcal{L}[\varepsilon_0] \tag{2b}$$

And, taking $\mathcal{L}[\sigma] = E_{0_K} \cdot \mathcal{L}[\varepsilon_0]$ one has:

$$\mathcal{L}[\varepsilon] = \left(\frac{1}{E_{0_K}} + \frac{1}{E_{1_K}+\eta_{1_K} \cdot s}\right) \cdot \mathcal{L}[\sigma] \tag{3}$$

Then, the mechanical resistance *Z*, the transfer function of the system, considering the stress as the input and the strain as the output is:

$$Z(s) = \frac{1}{E_{0_K}} + \frac{1}{E_{1_K}+\eta_{1_K} \cdot s} \tag{4}$$

Also from expression (3), the differential equation associated to the GKV model with *n*=1 is:

$$\sigma + \frac{\eta_{1K}}{E_{0K}+E_{1K}} \cdot \frac{d\sigma}{dt} = \frac{E_{0K}E_{1K}}{E_{0K}+E_{1K}} \varepsilon + \frac{E_{0K}\eta_{1K}}{E_{0K}+E_{1K}} \cdot \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{5}$$

The relaxation modulus is the stress response for a constant strain. This is $\mathcal{L}[\varepsilon] = 1/s$, in the Laplace domain. Therefore, the relaxation modulus is the transfer function from strain to stress. This is the inverse of the product between the mechanical resistance and *s*.

Through conversion from Laplace domain to time domain, the relaxation modulus for the GKV model with $n = 1$ gives:

$$Y = Y_0\left(1 - p_1\left(1 - \mathrm{e}^{-\frac{t}{\tau_1}}\right)\right) \tag{6a}$$

where

$$Y_0 = G_0 = E_{0_K}\,;\, p_1 = \frac{E_{0_K}}{E_{0_K}+E_{1_K}}\,;\, \tau_1 = \frac{\eta_{1_K}}{E_{0_K}+E_{1_K}} \tag{6b}$$

The creep compliance in the Laplace domain is the strain of the material under a constant unitary stress: $\sigma = 1\,, t > 0$. This means $\mathcal{L}[\sigma] = 1/s$. Therefore, $Y(s)\cdot J(s) = s^{-2}$, as it is pointed out in [37]. From (3), and converting from Laplace domain to time domain, the creep compliance J for the GKV model with $n = 1$ is:

$$J = J_0\left(1 + q_1\left(1 - \mathrm{e}^{-\frac{t}{\lambda_1}}\right)\right) \tag{7a}$$

where

$$J_0 = \frac{1}{E_{0_K}}\,;\, q_1 = \frac{E_{0_K}}{E_{1_K}}\,;\, \lambda_1 = \frac{\eta_{1_K}}{E_{1_K}} \tag{7b}$$

The frequency response function can be easily found from the expression of the mechanical resistance (4) just changing the s parameter by $\mathrm{i}\omega$, where "i" is the imaginary unit. After algebraic operation, the result is a complex function of ω. This function of the angular frequency ω describes the response (strain) of the material under harmonic stresses. The storage compliance (J') is the real part of this number.

$$J' = J_0\left(1 + \frac{q_1}{1+\lambda_1^2\omega^2}\right) \tag{8a}$$

The loss compliance (J'') is the absolute value of the imaginary part.

$$J'' = J_0\frac{q_1\cdot\lambda_1\,\omega}{1+\lambda_1^2\omega^2} \tag{8b}$$

Storage and loss moduli, in the frequency domain, are defined as the real and imaginary part of the inverse of the creep compliance. This is the response (stress) of the material under harmonic strains. The storage modulus is:

$$G' = G_0\left(1 - \frac{p_1}{1+\tau_1^2\omega^2}\right) \tag{9a}$$

The loss modulus is:

$$G'' = G_0\frac{p_1 \cdot \tau_1\,\omega}{1+\tau_1^2\omega^2} \tag{9b}$$

The tangent of the phase angle is the ratio of loss modulus (9b) to storage modulus (9a):

$$\tan\delta = \frac{\frac{p_1\,\tau_1\,\omega}{1+\tau_1^2\omega^2}}{1-\frac{p_1}{1+\tau_1^2\omega^2}} \tag{10}$$

1.1.2 Second order generalized Kelvin-Voigt model

Adding a new Kelvin-Voigt element, the GKV model with n = 2 (second order) has the representation of figure 3.

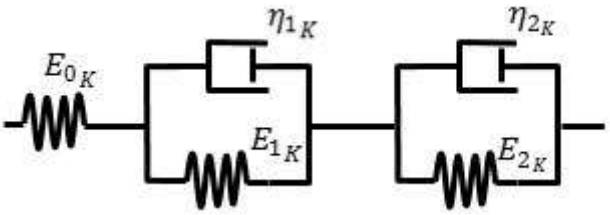

Figure 3 - Generalized Kelvin-Voigt with n = 2

Following a deduction analogous to that of the previous section, for the mechanical resistance, the GKV model with n = 2 gives:

$$Z = \frac{1}{E_{0K}} + \frac{1}{E_{1K}+\eta_{1K}\cdot s} + \frac{1}{E_{2K}+\eta_{2K}\cdot s} \tag{11}$$

The differential equation for the GKV model with n = 2, using Laplace transform, presents the following form:

$$\sigma + \frac{E_{0_K}(\eta_{1_K}+\eta_{2_K})+E_{1_K}\eta_{2_K}+E_{2_K}\eta_{1_K}}{E_{0_K}E_{1_K}+E_{0_K}E_{2_K}+E_{1_K}E_{2_K}} \cdot \frac{\mathrm{d}\sigma}{\mathrm{d}t} + \frac{\eta_{1_K}\eta_{2_K}}{E_{0_K}E_{1_K}+E_{0_K}E_{2_K}+E_{1_K}E_{2_K}} \cdot \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} =$$

$$\frac{E_{0_K}E_{1_K}E_{2_K}}{E_{0_K}E_{1_K}+E_{0_K}E_{2_K}+E_{1_K}E_{2_K}}\varepsilon + \frac{E_{0_K}(E_{1_K}\eta_{2_K}+E_{2_K}\eta_{1_K})}{E_{0_K}E_{1_K}+E_{0_K}E_{2_K}+E_{1_K}E_{2_K}} \cdot \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{E_{0_K}\eta_{1_K}\eta_{2_K}}{E_{0_K}E_{1_K}+E_{0_K}E_{2_K}+E_{1_K}E_{2_K}} \cdot \frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} \quad (12)$$

The creep compliance is $Z(s)/s$, which is, for the time domain:

$$J = J_0\left(1 + q_1\left(1 - \mathrm{e}^{-\frac{t}{\lambda_1}}\right) + q_2\left(1 - \mathrm{e}^{-\frac{t}{\lambda_2}}\right)\right) \quad (13)$$

where

$$J_0 = \frac{1}{E_{0_K}} \quad (14a)$$

$$q_\mathrm{i} = \frac{E_{0_K}}{E_{\mathrm{i}_K}} \ , i = 1{,}2 \quad (14b)$$

$$\lambda_\mathrm{i} = \frac{\eta_{\mathrm{i}_K}}{E_{\mathrm{i}_K}} \ , i = 1{,}2 \quad (14c)$$

Following the same procedure as explained to obtain the relaxation modulus in section 1.1.1, the relaxation modulus for the GKV model with n = 2 is defined as:

$$Y = Y_0\left(1 - p_1\left(1 - \mathrm{e}^{-\frac{t}{\tau_1}}\right) - p_2\left(1 - \mathrm{e}^{-\frac{t}{\tau_2}}\right)\right) \quad (15)$$

For this model, when analyzing the relaxation coefficients, these coefficients are found as a function of the auxiliary coefficients used in the previous expressions. Creep coefficients can be related to spring and dashpot parameters:

$$Y_0 = E_{0_K} \quad (16a)$$

$$p_1 = \frac{C_{2_K}(C_{4_K}+C_{5_K}-C_{3_K})}{2C_{4_K}(C_{1_K}+C_{2_K})} \quad (16b)$$

$$p_2 = \frac{C_{2_K}(C_{4_K}-C_{5_K}+C_{3_K})}{2C_{4_K}(C_{1_K}+C_{2_K})} \quad (16c)$$

$$\tau_1 = \frac{1}{C_{3_K}-C_{4_K}} \quad (16d)$$

$$\tau_2 = \frac{1}{C_{3K}+C_{4K}} \tag{16e}$$

where

$$C_{1K} = \frac{E_{0K}E_{1K}E_{2K}}{E_{0K}E_{1K}+E_{0K}E_{2K}+E_{1K}E_{2K}} \tag{17a}$$

$$C_{2K} = \frac{{E_{0K}}^2(E_{1K}+E_{2K})}{E_{0K}E_{1K}+E_{0K}E_{2K}+E_{1K}E_{2K}} \tag{17b}$$

$$C_{3K} = \frac{E_{0K}(\eta_{1K}+\eta_{2K})+E_{1K}\eta_{2K}+E_{2K}\eta_{1K}}{2\eta_{1K}\eta_{2K}} \tag{17c}$$

$$C_{4K} = \frac{\sqrt{(E_{0K}(\eta_{1K}+\eta_{2K})+E_{1K}\eta_{2K}+E_{2K}\eta_{1K})^2-4\eta_{1K}\eta_{2K}(E_{0K}E_{1K}+E_{0K}E_{2K}+E_{1K}E_{2K})}}{2\eta_{1K}\eta_{2K}} \tag{17d}$$

$$C_{5K} = \frac{{E_{1K}}^2\eta_{2K}+{E_{2K}}^2\eta_{1K}}{\eta_{1K}\eta_{2K}(E_{1K}+E_{2K})} \tag{17e}$$

Following the same procedure as explained to obtain the storage modulus in section 1.1.1, the storage modulus and the loss modulus are:

$$G' = G_0\left(1-\frac{p_1}{1+\tau_1^2\omega^2}-\frac{p_2}{1+\tau_2^2\omega^2}\right) \tag{18}$$

$$G'' = G_0\left(\frac{p_1\tau_1\omega}{1+\tau_1^2\omega^2}+\frac{p_2\tau_2\omega}{1+\tau_2^2\omega^2}\right) \tag{19}$$

where

$$G_0 = E_{0K} \tag{20}$$

The storage compliance and the loss compliance are, respectively:

$$J' = J_0\left(1+\frac{q_1}{1+\lambda_1^2\omega^2}+\frac{q_2}{1+\lambda_2^2\omega^2}\right) \tag{21}$$

$$J'' = J_0\left(\frac{q_1\lambda_1\omega}{1+\lambda_1^2\omega^2}+\frac{q_2\lambda_2\omega}{1+\lambda_2^2\omega^2}\right) \tag{22}$$

The tangent of the phase angle is the ratio of loss modulus (19) to storage modulus (18):

$$\tan\delta = \frac{\frac{p_1\tau_1\omega}{1+\tau_1^2\omega^2} + \frac{p_2\tau_2\omega}{1+\tau_2^2\omega^2}}{1-\left(\frac{p_1}{1+\tau_1^2\omega^2}\right)-\left(\frac{p_2}{1+\tau_2^2\omega^2}\right)} \tag{23}$$

1.1.3 Generalized Kelvin-Voigt model with *n* elements

This subsection is devoted to the general case (Fig. 1) with *n* Kelvin-Voigt elements. The elastic (E) and viscous (η) parameters are defined as usual (1). The stress along the model is the same for each block in figure 3, and the total strain of the model results of the strain summation of every block.

$$\varepsilon = \varepsilon_0 + \sum_{i=1}^{n} \varepsilon_i \tag{24a}$$

$$\sigma = \sigma_i, \quad i = 0 \dots n \tag{24b}$$

For every block, i = 1… n, the stress is:

$$\sigma = E_{0_K} \cdot \varepsilon_0 = E_{i_K} \cdot \varepsilon_i + \eta_{i_K} \cdot \frac{\mathrm{d}\varepsilon_i}{\mathrm{d}t} \; ; \quad i = 1 \dots n \tag{25}$$

Taking Laplace transformation of expressions (24-25) and combining them, one can define the mechanical resistance Z, the transfer function of the system (considering the stress as the input and the strain as the output).

$$Z(s) = \frac{\mathcal{L}[\varepsilon]}{\mathcal{L}[\sigma]} = \frac{1}{E_{0_K}} + \sum_{i=1}^{n} \frac{1}{E_{i_K}+\eta_{i_K}\cdot s} \tag{26}$$

Coming back to time domain, one can obtain the following differential equation:

$$\sigma + \frac{B}{A}\frac{d\sigma}{dt} + \frac{C}{A}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + \frac{D}{A}\frac{\mathrm{d}^3\sigma}{\mathrm{d}t^3} + \cdots + \frac{Y}{A}\frac{\mathrm{d}^{n-1}\sigma}{\mathrm{d}t^{n-1}} + \frac{Z}{A}\frac{\mathrm{d}^n\sigma}{\mathrm{d}t^n} =$$

$$\frac{a}{A}\varepsilon + \frac{b}{A}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{c}{A}\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} + \frac{d}{A}\frac{\mathrm{d}^3\varepsilon}{\mathrm{d}t^3} + \cdots + \frac{y}{A}\frac{\mathrm{d}^{n-1}\varepsilon}{\mathrm{d}t^{n-1}} + \frac{z}{A}\frac{\mathrm{d}^n\varepsilon}{\mathrm{d}t^n} \tag{27}$$

The constant values in (27) can be found in Appendix A.1.

The creep compliance in the Laplace domain is the strain of the material under a constant unitary stress: $\sigma = 1\,, t > 0$. This means $\mathcal{L}[\sigma] = 1/s$. From (26), and converting from Laplace domain to time domain, the creep compliance *J* for the GKV model results:

$$J = J_o(1 + \sum_{i=1}^{n} q_i \cdot [1 - \exp(-t/\lambda_i)]) \tag{28}$$

where

$$J_0 = \frac{1}{E_{0_K}} \tag{29a}$$

$$q_i = \frac{E_{0_K}}{E_{i_K}} \tag{29b}$$

$$\lambda_i = \frac{\eta_{i_K}}{E_{i_K}} \tag{29c}$$

As in previous sections, the frequency response function can be found from the expression of the mechanical resistance (26). The storage compliance (*J'*) is the real part of this number.

$$J' = J_0 \left(1 + \sum_{i=1}^{n} \frac{q_i}{1+\lambda_i^2 \omega^2}\right) \tag{30}$$

The loss compliance (*J"*) is the absolute value of the imaginary part.

$$J'' = J_0 \sum_{i=1}^{n} \frac{q_i \lambda_i \cdot \omega}{1+\lambda_i^2 \cdot \omega^2} \tag{31}$$

1.2 Generalized Maxwell model

A Maxwell element is a set of one spring and one dashpot connected in series. The generalized Maxwell model (also known as Maxwell-Wiechert model) is a set of *n* Maxwell elements with a spring, all them connected in parallel (figure 4).

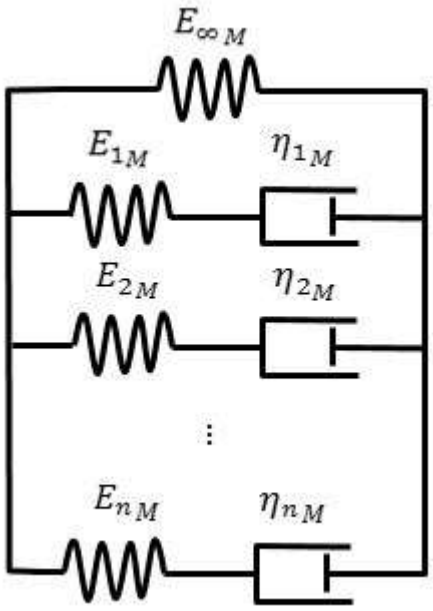

Figure 4: Generalized Maxwell model.

The elastic (E) and viscous (η) components are defined as usual (1). The strain along the model is the same for each block in figure 4, and the total stress of the model results of the stress summation of every block.

1.2.1 First order generalized Maxwell model

The generalized Maxwell model with n = 1 has the representation of figure 5. It is also known as Zener model.

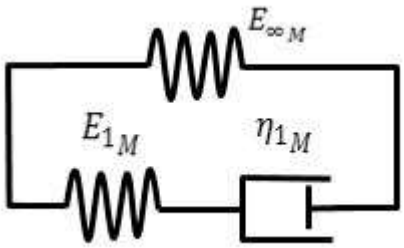

Figure 5: GM model n = 1.

The mechanical resistance Z is the transfer function of the system, considering the stress as the input and the strain as the output.

$$Z(s) = \left(E_{\infty_M} + \frac{E_{1_M}\eta_{1_M}\cdot s}{E_{1_M} + \eta_{1_M}\cdot s}\right)^{-1} \tag{32}$$

The differential equation for the GM model with n = 1, leads to:

$$\sigma + \frac{\eta_{1_M}}{E_{1_M}}\cdot\frac{\mathrm{d}\sigma}{\mathrm{d}t} = E_{\infty_M}\cdot\varepsilon + \frac{\eta_{1_M}}{E_{1_M}}\left(E_{\infty_M} + E_{1_M}\right)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{33}$$

As in the previous section, the relaxation modulus can be determined by the relationship $Z(s)\cdot Y(s) = 1/s$:

$$Y = Y_0\left(1 - p_1\left(1 - \mathrm{e}^{-\frac{t}{\tau_1}}\right)\right) \tag{34}$$

Also, the storage modulus and the loss modulus are the real and imaginary parts of $Y(i\omega)$.

$$G' = G_0\left(1 - \frac{p_1}{1+\tau_1^2\omega^2}\right) \tag{35}$$

$$G'' = G_0\frac{p_1\,\tau_1\,\omega}{1+\tau_1^2\omega^2} \tag{36}$$

where

$$Y_0 = G_0 = E_{\infty_M} + E_{1_M} \tag{37a}$$

$$p_1 = \frac{E_{1M}}{E_{\infty M}+E_{1M}} \tag{37b}$$

$$\tau_1 = \frac{\eta_{1M}}{E_{1M}} \tag{37c}$$

The tangent of the phase angle is the ratio of loss modulus (36) to storage modulus (35).

$$\tan\delta = \frac{\frac{p_1\tau_1\,\omega}{1+\tau_1^2\omega^2}}{1-\frac{p_1}{1+\tau_1^2\omega^2}} \tag{38}$$

Since $J(s) = Z(s)/s$ and $J(i\omega) = J'(\omega) + \mathrm{i}\cdot J''(\omega)$, the creep compliance, the storage compliance and the loss compliance are:

$$J = J_0\left(1 + q_1\left(1 - \mathrm{e}^{-\frac{t}{\lambda_1}}\right)\right) \tag{39}$$

$$J' = J_0\left(1 + \frac{q_1}{1+\lambda_1^2\omega^2}\right) \tag{40}$$

$$J'' = J_0 \cdot \frac{q_1\,\lambda_1\,\omega}{1+\lambda_1^2\,\omega^2} \tag{41}$$

where

$$J_0 = \frac{1}{E_{\infty M}+E_{1M}} \tag{42a}$$

$$q_1 = \frac{E_{1M}}{E_{\infty M}} \tag{42b}$$

$$\lambda_1 = \eta_{1M}\left(\frac{1}{E_{\infty M}} + \frac{1}{E_{1M}}\right) \tag{42c}$$

1.2.2 Second order generalized Maxwell model

The GM model with $n = 2$ has the representation of figure 6.

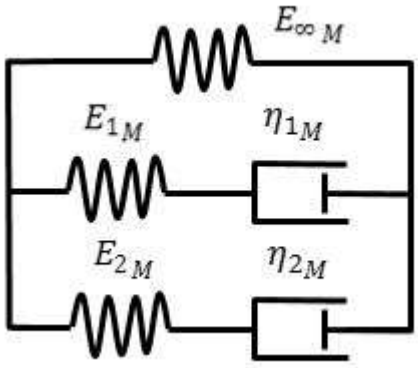

Figure 6: GM model $n = 2$.

The mechanical resistance of the GM model with $n = 2$ is:

$$Z = \left(E_{\infty M} + \frac{E_{1M}\eta_{1M}\cdot s}{E_{1M}+\eta_{1M}\cdot s} + \frac{E_{2M}\eta_{2M}\cdot s}{E_{2M}+\eta_{2M}\cdot s}\right)^{-1} \tag{43}$$

The differential equation for this model, using Laplace transform, shows the following expression:

$$\sigma + \frac{E_{1M}\eta_{2M}+E_{2M}\eta_{1M}}{E_{1M}E_{2M}}\cdot\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \frac{\eta_{1M}\eta_{2M}}{E_{1M}E_{2M}}\cdot\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} = E_{\infty M}\cdot\varepsilon + \left(\frac{E_{\infty M}}{E_{1M}E_{2M}}(E_{1M}\eta_{2M} + E_{2M}\eta_{1M}) + \eta_{1M} + \eta_{2M}\right)\cdot\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{\eta_{1M}\eta_{2M}}{E_{1M}E_{2M}}\left(E_{\infty M} + E_{1M} + E_{2M}\right)\cdot\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} \tag{44}$$

The relaxation modulus, the storage modulus and the loss modulus, are defined as follows:

$$Y = Y_0\left(1 - p_1\left(1 - \mathrm{e}^{-\frac{t}{\tau_1}}\right) - p_2\left(1 - \mathrm{e}^{-\frac{t}{\tau_2}}\right)\right) \tag{45}$$

$$G' = G_0\left(1 - \frac{p_1}{1+\tau_1^2\omega^2} - \frac{p_2}{1+\tau_2^2\omega^2}\right) \tag{46}$$

$$G'' = G_0\left(\frac{\mathrm{p}_1\tau_1\omega}{1+\tau_1^2\omega^2} + \frac{\mathrm{p}_2\tau_2\omega}{1+\tau_2^2\omega^2}\right) \tag{47}$$

where

$$Y_0 = G_0 = E_{\infty_M} + E_{1_M} + E_{2_M} \tag{48a}$$

$$p_{\mathrm{i}} = \frac{E_{\mathrm{i}_M}}{E_{\infty_M}+E_{1_M}+E_{2_M}} \quad ,\mathrm{i} = 1,2 \tag{48b}$$

$$\tau_{\mathrm{i}} = \frac{\eta_{\mathrm{i}_M}}{E_{\mathrm{i}_M}} \quad ,i = 1,2 \tag{48c}$$

For this model, when analyzing the creep coefficients, these coefficients are found as a function of auxiliary coefficients used in the previous expressions. Creep coefficients can be related to spring and dashpot parameters. The creep compliance, the storage compliance and the loss compliance are:

$$J = J_0\left(1 + q_1\left(1 - \mathrm{e}^{-\frac{t}{\lambda_1}}\right) + q_2\left(1 - \mathrm{e}^{-\frac{t}{\lambda_2}}\right)\right) \tag{49}$$

$$J' = J_0\left(1 + \frac{q_1}{1+\lambda_1^2\omega^2} + \frac{q_2}{1+\lambda_2^2\omega^2}\right) \tag{50}$$

$$J'' = J_0\left(\frac{q_1\lambda_1\omega}{1+\lambda_1^2\omega^2} + \frac{q_2\lambda_2\omega}{1+\lambda_2^2\omega^2}\right) \tag{51}$$

The tangent of the phase angle is the ratio of loss modulus (47) to storage modulus (46):

$$\tan\delta = \frac{\frac{\mathrm{p}_1\tau_1\omega}{1+\tau_1^2\omega^2} + \frac{\mathrm{p}_2\tau_2\omega}{1+\tau_2^2\omega^2}}{1-\frac{p_1}{1+\tau_1^2\omega^2} - \frac{p_2}{1+\tau_2^2\omega^2}} \tag{52}$$

with

$$J_0 = C_{1_M} + C_{2_M} \tag{53a}$$

$$q_1 = -\frac{C_{2_M}(C_{4_M}+C_{5_M}-C_{3_M})}{2C_{4_M}(C_{1_M}+C_{2_M})} \tag{53b}$$

$$q_2 = -\frac{C_{2_M}(C_{4_M}-C_{5_M}+C_{3_M})}{2C_{4_M}(C_{1_M}+C_{2_M})} \tag{53c}$$

$$\lambda_1 = \frac{1}{C_{3_M}-C_{4_M}} \tag{53d}$$

$$\lambda_2 = \frac{1}{C_{3_M}+C_{4_M}} \tag{53e}$$

where

$$C_{1_M} = \frac{1}{E_{\infty M}} \tag{54a}$$

$$C_{2_M} = -\frac{E_{1M}+E_{2M}}{E_{\infty M}(E_{\infty M}+E_{1M}+E_{2M})} \tag{54b}$$

$$C_{3_M} = \frac{E_{\infty M}(E_{1M}\eta_{2M}+E_{2M}\eta_{1M})+E_{1M}E_{2M}(\eta_{1M}+\eta_{2M})}{2\eta_{1M}\eta_{2M}(E_{\infty M}+E_{1M}+E_{2M})} \tag{54c}$$

$$C_{4_M} = \frac{\sqrt{\left(E_{\infty M}(E_{1M}\eta_{2M}+E_{2M}\eta_{1M})+E_{1M}E_{2M}(\eta_{1M}+\eta_{2M})\right)^2-4E_{\infty M}E_{1M}E_{2M}\eta_{1M}\eta_{2M}(E_{\infty M}+E_{1M}+E_{2M})}}{2\eta_{1M}\eta_{2M}(E_{\infty M}+E_{1M}+E_{2M})} \tag{54d}$$

$$C_{5_M} = \frac{E_{1M}E_{2M}(\eta_{1M}+\eta_{2M})}{\eta_{1M}\eta_{2M}(E_{1M}+E_{2M})} \tag{54e}$$

1.2.3 Generalized Maxwell model with *n* elements

The mechanical resistance of the GM model, in the frequency domain, is defined as follows:

$$Z = \left(E_{\infty M} + \sum_{i=1}^{n} \frac{E_{i_M}\eta_{i_M}\cdot s}{E_{i_M}+\eta_{i_M}\cdot s}\right)^{-1} \tag{55}$$

The differential equation of the GM model, using Laplace transforms, presents the following form:

$$\sigma + \frac{B}{A}\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \frac{C}{A}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + \frac{D}{A}\frac{\mathrm{d}^3\sigma}{dt^3} + \cdots + \frac{Y}{A}\frac{\mathrm{d}^{n-1}\sigma}{\mathrm{d}t^{n-1}} + \frac{Z}{A}\frac{\mathrm{d}^n\sigma}{\mathrm{d}t^n} =$$

$$\frac{a}{A}\varepsilon + \frac{b}{A}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{c}{A}\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} + \frac{d}{A}\frac{\mathrm{d}^3\varepsilon}{\mathrm{d}t^3} + \cdots + \frac{y}{A}\frac{\mathrm{d}^{n-1}\varepsilon}{\mathrm{d}t^{n-1}} + \frac{z}{A}\frac{\mathrm{d}^n\varepsilon}{\mathrm{d}t^n} \tag{56}$$

The values of the constants in (56) can be found in Appendix A.1.2.

The relaxation modulus, the storage modulus and the loss modulus for the GM model are defined by the next equations:

$$Y = Y_0\left(1 - \sum_{i=1}^{n} p_i\left(1 - e^{-\frac{t}{\tau_i}}\right)\right) \tag{57}$$

$$G' = G_0\left(1 - \sum_{i=1}^{n}\left(\frac{p_i}{1+\tau_i^2\omega^2}\right)\right) \tag{58}$$

$$G'' = G_0 \cdot \sum_{i=1}^{n} \frac{p_i\tau_i\omega}{1+\tau_i^2\omega^2} \tag{59}$$

where:

$$Y_0 = G_0 = E_{\infty_M} + \sum_{i=1}^{n} E_{i_M} \tag{60a}$$

$$p_i = \frac{E_{i_M}}{E_{\infty_M}+\sum_{i=1}^{n} E_{i_M}} \tag{60b}$$

$$\tau_i = \frac{\eta_{i_M}}{E_{i_M}} \tag{60c}$$

2 Plots

It is possible to plot the relaxation modulus vs time, or the creep compliance vs time, for every viscoelastic model. Also, it is possible to plot storage/loss modulus/compliance and phase angle in the frequency domain. These plots, for each viscoelastic material model, related to a differential equation (27, 56), have some characteristics that can be determined and can allow to recognize the kind of model one is dealing with.

It is worth noting that the plots for GKV and GM models with n = 1 and with n = 2 are the same, as a function of either relaxation or creep coefficients, regardless of whether they are subsequently expressed as a function of the creep or relaxation coefficients. The reason is that although the dispositions of springs and dashpots are different for each model, the general expression for moduli and compliances for this number of terms is identical, expressed as relaxation or creep coefficients, respectively.

In order to draw the plots, first of all the mechanical parameter in question is represented as a function of time or frequency. So, a general plot is represented without any fixed values in both axes. The coefficients of the equations are expressed in a general way.

Once the plot is represented, the maximums and minimums are found with the first order derivative of the mathematical expression, set equal to zero (appendix A.2.1.1 and A.2.2.1). The inflexion points are found with the second order derivative of the previous mathematical expression also set equal to zero (appendix A.2.1.2 and A.2.2.2).

Relaxation moduli and creep compliances have not either maximums, nor minimums or inflexion points. In these cases, the characteristic points are just at zero and at infinite time.

It is important to emphasize that, for models of order greater than two, the determination of the relative extrema and inflexion points depends on the obtaining of the roots of a polynomial of grade greater than four. This avoids getting analytical expressions for these models.

2.1 Models with $n = 1$

Figure 7 shows a sketch of the characteristic plots of relaxation modulus and creep compliance along the time (equations 6a, 7a, 34, 39). Both curves are monotonic. The initial values and the limit values for time tending to infinity are characteristics of the model.

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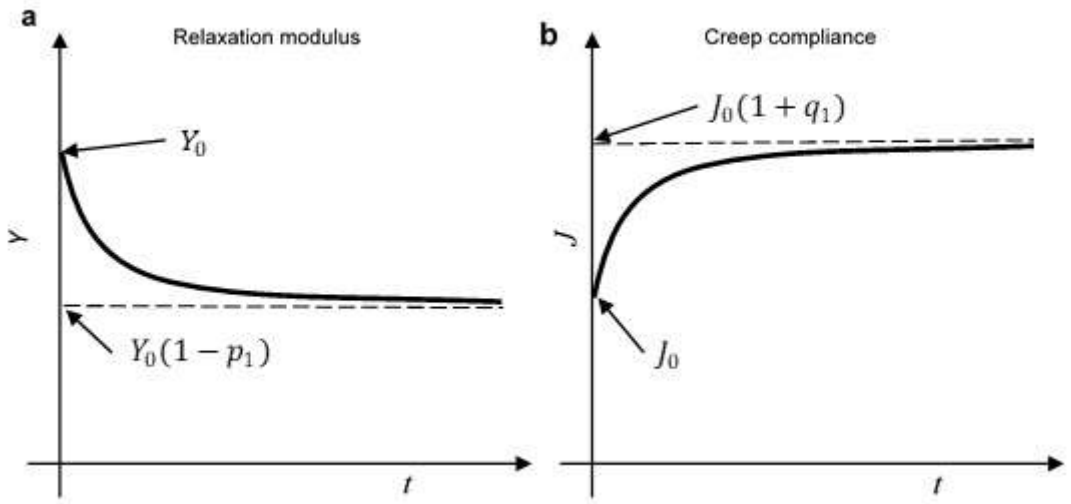

Figure 7: Relaxation modulus/creep compliance vs time: characteristic values. Models with $n = 1$.

Figure 8 (top, left) shows the plot of the storage modulus vs frequency (equations 9a, 35): there is one inflexion point and the limits at infinity are also determined. The plot of the loss modulus vs frequency (equations 9b, 36) has also one inflexion point and one maximum (Fig. 8, top, right). The loss modulus tends to zero when the frequency tends to infinity. The plots of the storage and loss compliances (equations 8a, 8b, 40, 41) have a quite similar structure (Fig. 8, bottom).

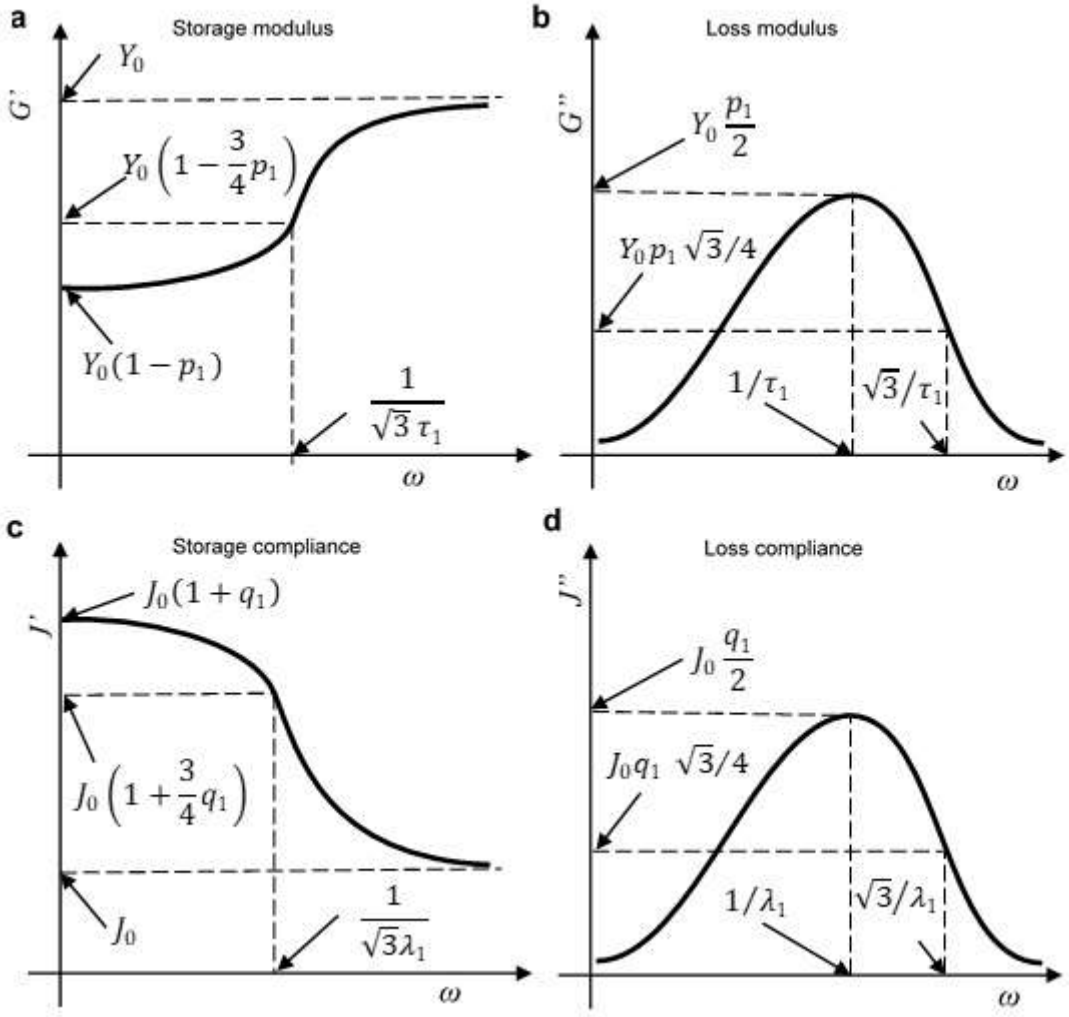

Figure 8: Storage/loss modulus (top) and compliance (bottom) vs frequency: characteristic values. Models with $n = 1$. The horizontal axes are in logarithmic scale.

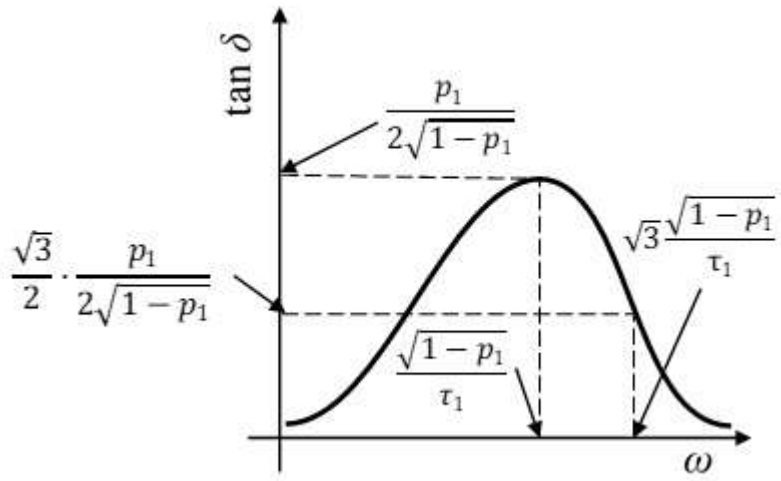

Figure 9: Phase angle vs frequency: characteristic values. Models with $n = 1$. The horizontal axes is in logarithmic scale.

Finally, figure 9 plots the phase angle along the frequency (equations 10, 38). This plot

has also a maximum and one inflexion point (and another theoretical one for frequency zero).

2.2 Plots with $n = 2$

The general plots between the different mechanical properties versus time or frequency are represented in figures 10-12, for the cases when there are two summations in the corresponding Prony series. Figure 10 shows the relaxation modulus and the creep compliance versus time (equations 13, 15, 45, 49). Both curves are monotonic and have a simple value when time tends to infinity.

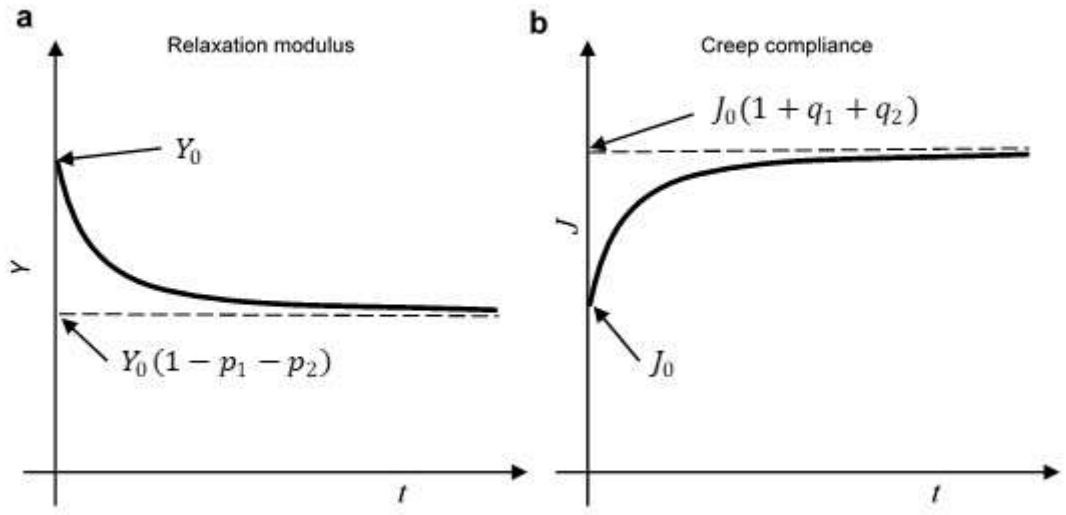

Figure 10: Relaxation modulus/creep compliance vs time: characteristic values. Models with $n = 2$.

From the expression of the storage modulus (equations 18, 46), one can calculate the limit values at frequency zero and tending to infinity. The function is strictly increasing so there are no relative extrema. There are three inflexion points (Fig. 11, top, left) that can be calculated by setting the second derivative of equations 18 and 46 equal to zero. This leads to the fourth grade equation (A.10 found in appendix A.2.2.2, set equal to zero).

$$x^4 + \frac{9\,\tau_1^4\tau_2^4(p_1+p_2)-p_1\tau_2^4+p_2\tau_1^4}{3\tau_1^2\tau_2^2(p_1\tau_2^2+p_2\tau_1^2)}x^3 + \frac{3(p_1\tau_1^2+p_2\tau_2^2)-p_1\tau_2^2-p_2\tau_1^2)}{\tau_1^2\tau_2^2(p_1\tau_2^2+p_2\tau_1^2)}x^2 + \frac{(\tau_1^2-\tau_2^2)(p_1\tau_1^2-p_2\tau_2^2)}{\tau_1^4\tau_2^4(p_1\tau_2^2+p_2\tau_1^2)}x -$$

$$\frac{p_1\tau_1^2+p_2\tau_2^2}{3\,\tau_1^4\tau_2^4(p_1\tau_2^2+p_2\tau_1^2)} = 0 \qquad (61)$$

where $x = \omega^2$. This equation can be solved in explicit form. The resulting frequencies for the inflexion points are (appendix A.3.2):

$$\omega_{\text{infl}\,i} = \sqrt{x_{i+1}} \quad , i = 1,2,3. \qquad (62a)$$

$$G'_{\text{infl}\,i} = G_0\left(1 + \frac{p_1}{1+\tau_1^2\omega_{\text{infl}\,i}^2} + \frac{p_2}{1+\tau_2^2\omega_{\text{infl}\,i}{}^2}\right) \qquad (62b)$$

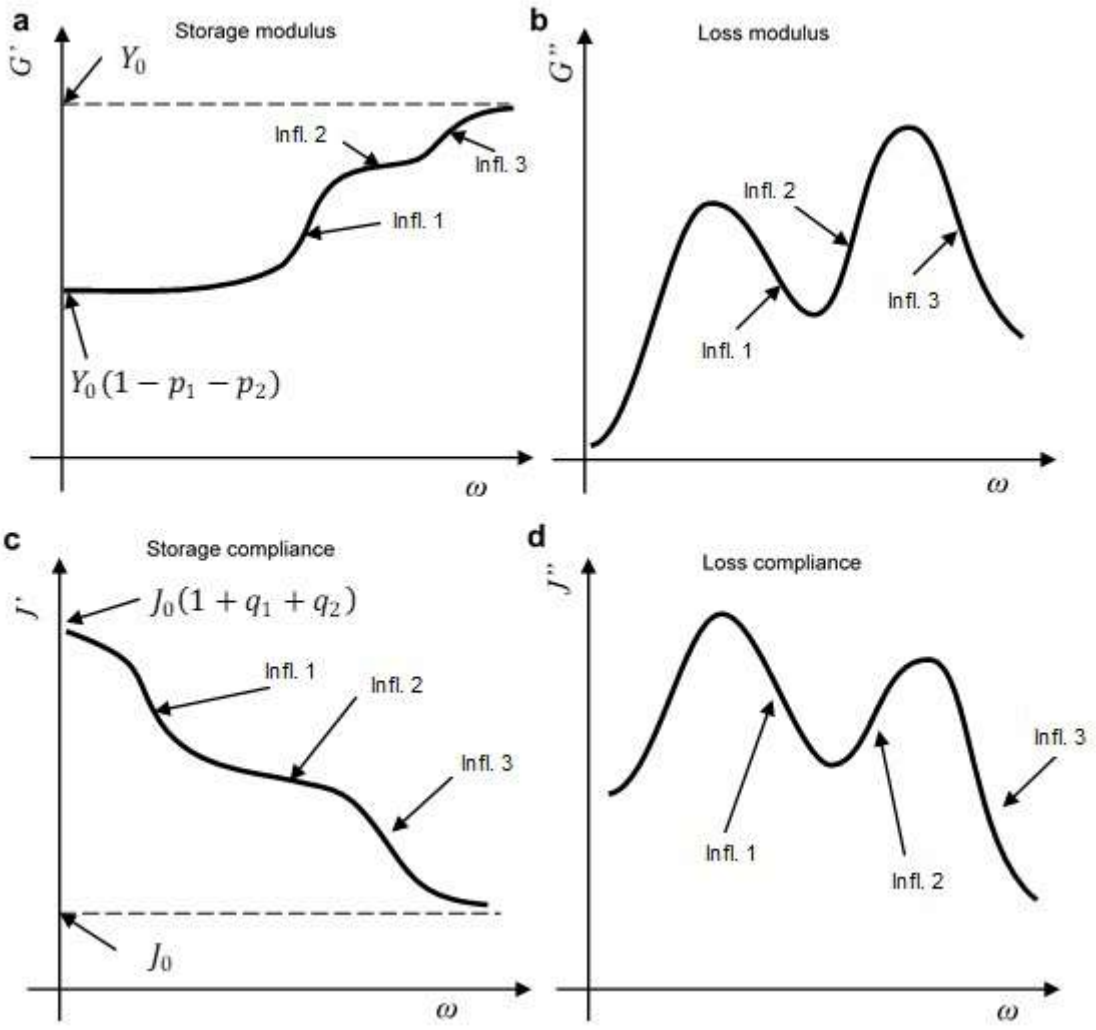

Figure 11: Sketches of typical graphs of storage/loss modulus (top) and compliance (bottom) vs frequency for models with n = 2. The arrows mark the inflexion points. The horizontal axes are in logarithmic scale.

The loss modulus (figure 11, top, right) has two maximums and one minimum, with three inflexions points (equation A.6 in appendix A.2.2.1). The maximums and the minimum can be localized by deriving the expression (19, 47) and their frequencies are the roots of the cubic equation:

$$x^3 + \frac{p_1\tau_1\tau_2^2(2\tau_1^2-\tau_2^2)+p_2\tau_2\tau_1^2(2\tau_2^2-\tau_1^2)}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)}x^2 + \frac{p_1\tau_1(\tau_1^2-2\tau_2^2)+p_2\tau_2(\tau_2^2-2\tau_1^2)}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)}x - \frac{p_1\tau_1+p_2\tau_2}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)} = 0 \quad (63)$$

where $x = \omega^2$. This equation can be solved in explicit form. The resulting frequencies are (appendix A.3.1):

$$\omega_{\max 1} = \sqrt{x_1} \; < \; \omega_{\max 2} = \sqrt{x_3} \quad (64a)$$

$$G''_{\max i} = G_0\left(\frac{p_1\tau_1\omega_{\max i}}{1+\tau_1^2\omega_{\max i}^2} + \frac{p_2\tau_2\omega_{\max i}}{1+\tau_2^2\omega_{\max i}^2}\right) \; , i = 1,2 \quad (64b)$$

$$\omega_{\min} = \sqrt{x_2} \quad (64c)$$

$$G''_{\min} = G_0\left(\frac{p_1\tau_1\omega_{\min}}{1+\tau_1^2\omega_{\min}^2} + \frac{p_2\tau_2\omega_{\min}}{1+\tau_2^2\omega_{\min}^2}\right) \quad (64d)$$

From (19, 47), using equation A.11 (in appendix A.2.2.2) set equal to zero, where $x = \omega^2$, one obtains the following expression:

$$x^4 + \frac{3\left((\tau_1^2-\tau_2^2)(p_1\tau_2-p_2\tau_1)\right)}{\tau_1^2\tau_2^2(p_1\tau_2+p_2\tau_1)}x^3 + \frac{3\left((p_1\tau_1^3+\tau_2^3)-3\tau_1\tau_2(p_1\tau_2+p_2\tau_1)\right)}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)}x^2 +$$

$$\frac{p_1\tau_1^5+p_2\tau_2^5-9\,\tau_1^2\tau_2^2(p_1\tau_1+p_2\tau_2)}{\tau_1^5\tau_2^5(p_1\tau_2+p_2\tau_1)}x - 3\frac{p_1\tau_1^3+p_2\tau_2^3}{\tau_1^5\tau_2^5(p_1\tau_2+p_2\tau_1)} = 0 \quad (65)$$

The inflexion points correspond to the roots of equation 65. This leads (see appendix A.3.2) to the next frequencies for the three inflexion points (Fig. 11, top, right):

$$\omega_{G''\text{infl }1} = \sqrt{x_2} \; ; \; \omega_{G''\text{infl }2} = \sqrt{x_3} \; ; \; \omega_{G''\text{infl }3} = \sqrt{x_4} \quad (66a)$$

$$G''(\omega_{G''\text{infl }i}) = G_0\left(\frac{p_1\tau_1\omega_{G''\text{infl }i}}{1+{\tau_1}^2{\omega_{G''\text{infl }i}}^2} + \frac{p_2\tau_2\omega_{G''\text{infl }i}}{1+{\tau_2}^2{\omega_{G''\text{infl }i}}^2}\right) \quad ; \; i = 1,2,3 \quad (66b)$$

where the roots of $x_i, i = 2,3,4$ have explicit expressions (equations A.20-22); note that the first root, x_1 (A.19) is always negative. Figure 11 (bottom, left), shows the sketch of a typical graph for the storage compliance. This graph is defined by three inflexion points. By deriving (21, 50) twice (A.12, in appendix A.2.2.2), and setting equal to zero, the following polynomial equation is obtained:

$$x^4 + \left(\frac{9\lambda_1^4\lambda_2^4(q_1+q_2)-q_1\lambda_2^4+q_2\lambda_1^4}{3\lambda_1^2\lambda_2^2(q_1\lambda_2^2+q_2\lambda_1^2)}\right)x^3 + \frac{3(q_1\lambda_1^2+q_2\lambda_2^2)-q_1\lambda_2^2-q_2\lambda_1^2)}{\lambda_1^2\lambda_2^2(q_1\lambda_2^2+q_2\lambda_1^2)}x^2 + \frac{(\lambda_1^2-\lambda_2^2)(q_1\lambda_1^2-q_2\lambda_2^2)}{\lambda_1^4\lambda_2^4(q_1\lambda_2^2+q_2\lambda_1^2)}x - \frac{q_1\lambda_1^2+q_2\lambda_2^2}{3\lambda_1^4\lambda_2^4(q_1\lambda_2^2+q_2\lambda_1^2)} = 0 \quad (67)$$

where $x = \omega^2$. Solving this equation (see appendix A.3.2), the inflexion points of the graph are:

$$\omega_{J'\text{infl }1} = \sqrt{x_2}\,;\ \omega_{J'\text{infl }2} = \sqrt{x_3}\,;\omega_{J'\text{infl }3} = \sqrt{x_4} \quad (68a)$$

$$J'\left(\omega_{J'\text{infl }i}\right) = J_0\left(1 + \frac{q_1}{1+\lambda_1{}^2\omega_{J'\text{infl }i}{}^2} + \frac{q_2}{1+\lambda_2{}^2\omega_{J'\text{infl }i}{}^2}\right) \quad ; i = 1,2,3 \quad (68b)$$

Figure 11 (bottom, right) is the loss compliance vs frequency. The curve has two maximums, one minimum and three inflexion points. Deriving the equations 22 and 51 (see A.8, appendix A.2.2.1), and setting equal to zero, the resulting polynomial equation is as follows:

$$x^3 + \frac{q_1\lambda_2(2\lambda_1^2-\lambda_2^2)+q_2\lambda_1(2\lambda_2^2-\lambda_1^2)}{\lambda_1^2\lambda_2^2(q_1\lambda_2+q_2\lambda_1)}x^2 + \frac{q_1\lambda_1(\lambda_1^2-2\lambda_2^2)+q_2\lambda_2(\lambda_2^2-2\lambda_1^2)}{\lambda_1^3\lambda_2^3(q_1\lambda_2+q_2\lambda_1)}x - \frac{q_1\lambda_1+q_2\lambda_2}{\lambda_1^3\lambda_2^3(q_1\lambda_2+q_2\lambda_1)} = 0 \quad (69)$$

where $x = \omega^2$. This allows finding the maximums and minimum. The resulting values are:

$$\omega_{J''\text{max1}} = \sqrt{x_1};\ \omega_{J''\text{min}} = \sqrt{x_2};\ \omega_{J''\text{max2}} = \sqrt{x_3} \quad (70a)$$

$$J''\left(\omega_{J''i}\right) = J_0\left(\frac{q_1\lambda_1\omega_{J''i}}{1+\lambda_1{}^2\omega_{J''i}{}^2} + \frac{q_2\lambda_2\omega_{J''i}}{1+\lambda_2{}^2\omega_{J''i}{}^2}\right) ; i = \text{max1}, \text{min}, \text{max2} \quad (70b)$$

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En: *Acta Mechanica Sinica*, 2019. Vol.35 no.6, pp.1191-1209.
Disponible a: https://doi.org/10.1007/s10409-019-00895-6

With a second derivation of equations 22 and 51, (A.13, appendix A.2.2.2) one obtains:

$$x^4 + \frac{3\left((\lambda_1^2-\lambda_2^2)(q_1\lambda_2-q_2\lambda_1)\right)}{\lambda_1^2\lambda_2^2(q_1\lambda_2+q_2\lambda_1)}x^3 + \frac{3\left((q_1\lambda_1^3+q_2\lambda_2^3)-3\lambda_1\lambda_2(q_1\lambda_2+q_2\lambda_1)\right)}{\lambda_1^3\lambda_2^3(q_1\lambda_2+q_2\lambda_1)}x^2 +$$

$$\frac{q_1\lambda_1^5+q_2\lambda_2^5-9\lambda_1^2\lambda_2^2(q_1\lambda_1+q_2\lambda_2)}{\lambda_1^5\lambda_2^5(q_1\lambda_2+q_2\lambda_1)}x - 3\frac{q_1\lambda_1^3+q_2\lambda_2^3}{\lambda_1^5\lambda_2^5(q_1\lambda_2+q_2\lambda_1)} = 0 \qquad (71)$$

where $x = \omega^2$. The inflexion points are:

$$\omega_{J''\text{infl}1} = \sqrt{x_2}\ ;\ \omega_{J''\text{infl}2} = \sqrt{x_3}\ ;\ \omega_{J''\text{infl}3} = \sqrt{x_4} \qquad (72a)$$

$$J''\left(\omega_{J''\text{infl}\,i}\right) = J_0\left(\frac{q_1\lambda_1\omega_{J''\text{infl}\,i}}{1+\lambda_1^2\omega^2_{J''\text{infl}\,i}} + \frac{q_2\lambda_2\omega_{J''\text{infl}\,i}}{1+\lambda_2^2\omega^2_{J''\text{infl}\,i}}\right)\ ;i = 1,2,3 \qquad (72b)$$

The typical curve of the tangent of the phase angle vs frequency has also two maximums, one minimum and three inflexion points (figure 12). The first derivative of equations 23 and 52 leads to the polynomic equation (where $x = \omega^2$):

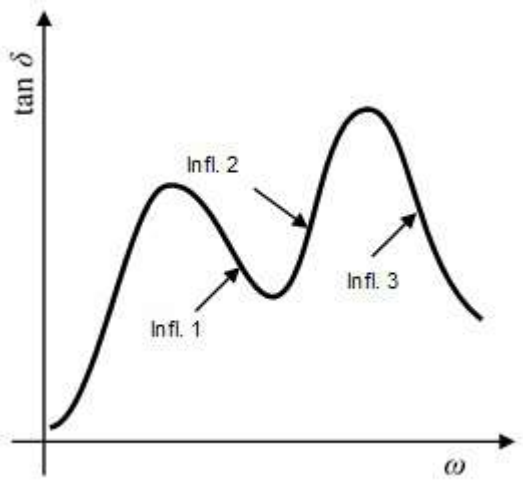

Figure 12: sketch of a typical graph of the tangent of phase angle vs frequency for models with *n* = 2. The arrows mark the inflexion points. The horizontal axes are in logarithmic scale.

$$x^3 + \frac{p_1\tau_2\left(2\tau_1^2-\tau_2^2(1-p_1)\right)+p_2\tau_1\left(2\tau_2^2-\tau_1^2(1-p_2)\right)+p_1p_2\tau_1\tau_2(\tau_1+\tau_2)}{\tau_1^2\tau_2^2(p_1\tau_2+p_2\tau_1)}x^2 +$$

$$\frac{p_1\tau_1^3+p_2\tau_2^3-2\tau_1\tau_2\left(p_1\tau_2(1-p_1)+p_2\tau_1(1-p_2)\right)+p_1p_2(\tau_1^2(3\tau_2-\tau_1)+\tau_2^2(3\tau_1-\tau_2))}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)}x +$$

$$\frac{p_1p_2(\tau_1+\tau_2)-p_1\tau_1(1-p_1)-p_2\tau_2(1-p_2)}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)} = 0 \qquad (73)$$

The maximums and minimum are:

$$\omega_{\tan\delta\,\max1} = \sqrt{x_1};\ \omega_{\tan\delta\,\min} = \sqrt{x_2};\ \omega_{\tan\delta\,\max2} = \sqrt{x_3} \qquad (74a)$$

$$\tan\delta\left(\omega_{\tan\delta\,i}\right) = \left(\frac{\frac{p_1\tau_1\omega_{\tan\delta\,i}}{1+\tau_1^2\omega_{\tan\delta\,i}^2}+\frac{p_2\tau_2\omega_{\tan\delta\,i}}{1+\tau_2^2\omega_{\tan\delta\,i}^2}}{1-\frac{p_1}{1+\tau_1^2\omega_{\tan\delta\,i}^2}-\frac{p_2}{1+\tau_2^2\omega_{\tan\delta\,i}^2}}\right);\ i = \max1, \min, \max2 \qquad (74b)$$

The second derivative of equations 23 and 52 gives the following equation:

$$x^4 + \frac{3(\tau_1\tau_2\left(p_1\tau_1(1+p_2)+p_2\tau_2(1+p_1)\right)-\tau_1^3p_2(1-p_2)-\tau_2^3p_1(1-p_1))}{\tau_1^2\tau_2^2(p_1\tau_2+p_2\tau_1)}x^3 +$$

$$3\frac{{\tau_1}^3p_1(1-p_2)+{\tau_2}^3p_2(1-p_1)+\tau_1\tau_2\left(p_1\tau_2(4p_1+3p_2-3)+p_2\tau_1(4p_2+3p_1-3)\right)}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)}x^2 +$$

$$\left[\frac{\tau_1^5p_1(1-p_2)^2+\tau_2^5p_2(1-p_1)^2+p_1p_2\tau_1\tau_2(\tau_1^3(1-p_2)+\tau_2^3(1-p_1))}{\tau_1^5\tau_2^5(p_1\tau_2+p_2\tau_1)}+\right.$$

$$\left.\frac{p_1\tau_1\left(p_2(10+p_1-p_2)-9(1-p_1)\right)+p_2\tau_2\left(p_1(10+p_2-p_1)-9(1-p_2)\right)}{\tau_1^3\tau_2^3(p_1\tau_2+p_2\tau_1)}\right]x +$$

$$3\frac{(p_1+p_2-1)\left((p_1\tau_1^3(1-p_2)+p_2\tau_2^3(1-p_1))+p_1p_2\tau_1\tau_2(\tau_1+\tau_2)\right)}{\tau_1^5\tau_2^5(p_1\tau_2+p_2\tau_1)} = 0 \qquad (75)$$

The inflexion points are:

$$\omega_{\tan\delta\,\text{infl}1} = \sqrt{x_2}\ ;\ \omega_{\tan\delta\,\text{infl}2} = \sqrt{x_3}\ ;\ \omega_{\tan\delta\,\text{infl}3} = \sqrt{x_4} \qquad (76a)$$

$$\tan\delta\left(\omega_{\tan\delta\,\text{infl}\,i}\right) = \left(\frac{\frac{p_1\tau_1\omega_{\tan\delta\,\text{infl}\,i}}{1+\tau_1^2\omega_{\tan\delta\,\text{infl}\,i}^2}+\frac{p_2\tau_2\omega_{\tan\delta\,\text{infl}\,i}}{1+\tau_2^2\omega_{\tan\delta\,\text{infl}\,i}^2}}{1-\frac{p_1}{1+\tau_1^2\omega_{\tan\delta\,\text{infl}\,i}^2}-\frac{p_2}{1+\tau_2^2\omega_{\tan\delta\,\text{infl}\,i}^2}}\right);\ i = 1,2,3 \qquad (76b)$$

3 Relationships between generalized Kelvin-Voigt and Maxwell models and relaxation and creep coefficients.

The equivalence of generalized Kelvin-Voigt and Maxwell models of the same order is well known. Nevertheless, unfortunately, it is not easy to find the algebraic expressions of this equivalence. In this section, these expressions are determined and presented in a closed form, up to second order models. Moreover, the relaxation coefficients for the moduli equations and the creep coefficients for the compliance equations are also found. Furthermore, explicit formulas are presented for the parameters of the material models (GKV and GM) from relaxation or creep coefficients. These relationships, up to second order models, can be found in Appendix B.1 and B.2.

In the diagram of figure 13, an application of the formulas developed in this work is shown, by way of example. In this case, starting from a Dynamic mechanical analysis (DMA) test, the parameters of the relaxation model can be obtained. This is carried out using the equations and their corresponding expressions in appendix A. From these relaxation parameters, by means of the interconversion expressions of the table shown in appendix B, the parameters corresponding to the creep test can be obtained. In addition, with the same table, the parameters of the material models are obtained, whether they are of the generalized Maxwell model or the generalized Kelvin-Voigt model.

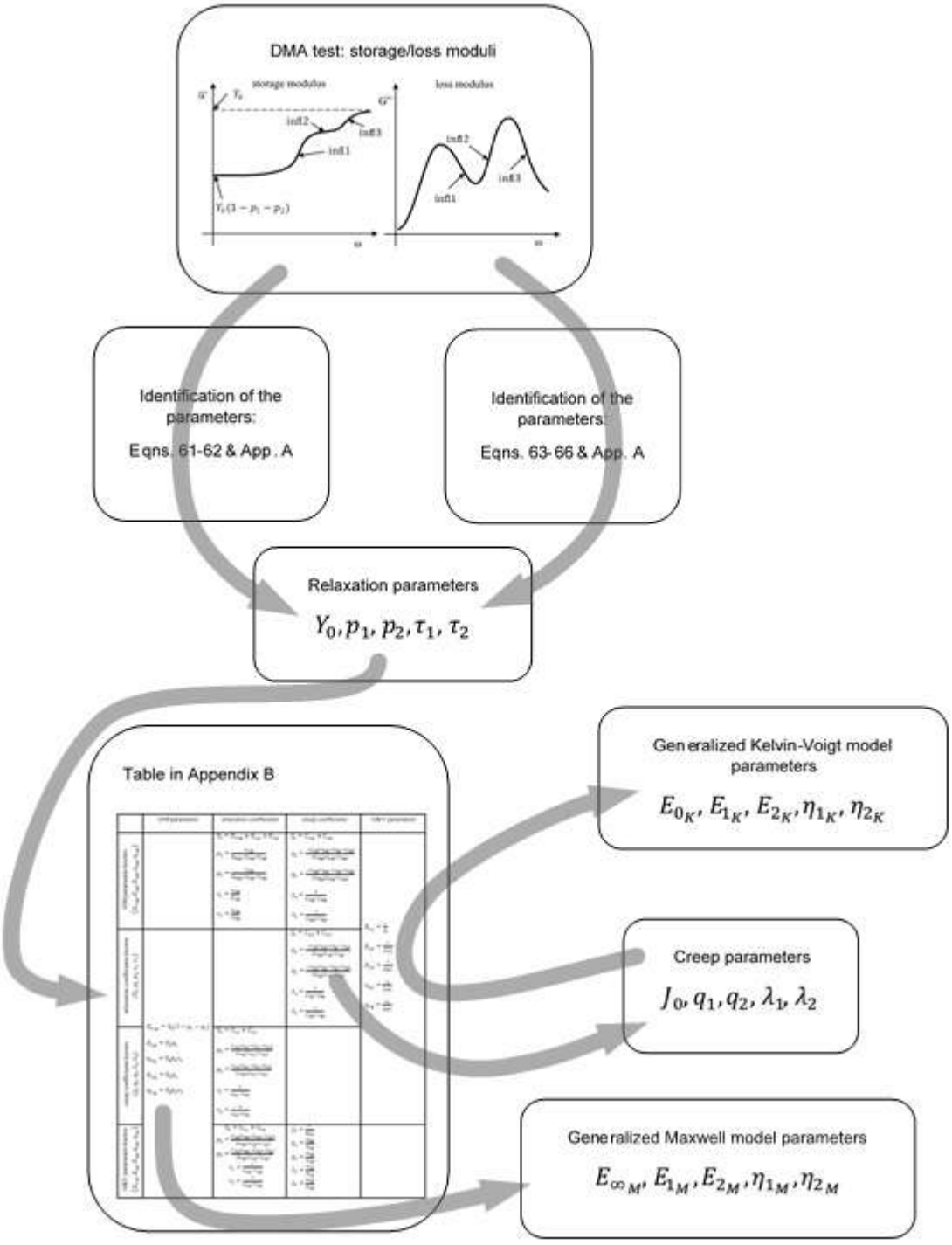

Figure 13 – Flowchart of an application: obtaining the creep and relaxation parameters, as well as the GM and GKV material models, from a DMA test.

4 Conclusions

The GKV and GM models of the same order are related to each other. These relationships can be helpful for the practitioner engineer, because computational simulators are implemented with just some specific combinations of springs and dashpots. The explicit relationships between the GKV and GM models, up to order two, are found. Moreover, explicit formulas are found for the position of the characteristic points (maxima, minima and inflexion points) of the storage and loss compliances/moduli.

The interconversion formulas for models of order 1 and 2 have also been developed. The complete set of interconversion formulas between GKV, GM, relaxation coefficients and creep coefficients have been presented.

Therefore, the practitioner engineer can easily have a simple guide to change from one model to another if necessary.

Appendix A

A.1 Constants of differential equations.

A.1.1 Constants of equation (27) in the text

$$A = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} E_{i_K} = \Delta \cdot \sum_{i=0}^{n} \frac{1}{E_{i_K}}$$

where $\Delta = \prod_{i=0}^{n} E_{i_K}$

$$B = \frac{\Delta}{E_{0_K}} \cdot \sum_{j=1}^{n} \frac{\eta_{j_K}}{E_{j_K}} + \Delta \cdot \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\eta_{j_K}}{E_{i_K} E_{j_K}}$$

$$C = \frac{\Delta}{E_{0_K}} \cdot \sum_{i=1}^{n-1} \sum_{\substack{j=2 \\ j>i}}^{n} \frac{\eta_{i_K} \eta_{j_K}}{E_{i_K} E_{j_K}} + \Delta \cdot \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \sum_{\substack{k=2 \\ k>j \\ k\neq i}}^{n} \frac{\eta_{j_K} \eta_{k_K}}{E_{i_K} E_{j_K} E_{k_K}}$$

$$D = \frac{\Delta}{E_{0_K}} \cdot \sum_{i=1}^{n-2} \sum_{\substack{j=2 \\ j>i}}^{n-1} \sum_{\substack{k=3 \\ k>j}}^{n} \frac{\eta_{i_K} \eta_{j_K} \eta_{k_K}}{E_{i_K} E_{j_K} E_{k_K}} + \Delta \cdot \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n-2} \sum_{\substack{k=2 \\ k>j \\ k\neq i}}^{n-1} \sum_{\substack{l=3 \\ l>k \\ l\neq i}}^{n} \frac{\eta_{j_K} \eta_{k_K} \eta_{l_K}}{E_{i_K} E_{j_K} E_{k_K} E_{k_K}}$$

$$\vdots$$

$$Y = \sum_{i=1}^{n} \left[(E_{0_K} + E_{i_K}) \prod_{\substack{j=1 \\ j\neq i}}^{n} \eta_{j_K} \right]$$

$$Z = \prod_{i=1}^{n} \eta_{i_K}$$

and

$$a = \Delta$$

$$b = \Delta \cdot \sum_{i=1}^{n} \frac{\eta_{i_K}}{E_{i_K}}$$

$$c = \Delta \cdot \sum_{i=1}^{n-1} \sum_{\substack{j=2 \\ j>i}}^{n} \frac{\eta_{i_K} \eta_{j_K}}{E_{i_K} E_{j_K}}$$

$$d = \Delta \cdot \sum_{i=1}^{n-2} \sum_{\substack{j=2 \\ j>i}}^{n-1} \sum_{\substack{k=3 \\ k>j}}^{n} \frac{\eta_{i_K} \eta_{j_K} \eta_{k_K}}{E_{i_K} E_{j_K} E_{k_K}}$$

$$\vdots$$

$$y = E_{0_K} \cdot \sum_{i=1}^{n} E_{i_K} \prod_{\substack{j=1 \\ j\neq i}}^{n} \eta_{j_K}$$

$$z = E_{0_K} \cdot \prod_{i=1}^{n} \eta_{i_K}$$

A.1.2 Constants of equation (56) in the text.

$$A = \prod_{i=1}^{n} E_{i_M} \equiv \Delta$$

$$B = \Delta \cdot \sum_{i=1}^{n} \frac{\eta_{i_M}}{E_{i_M}}$$

$$C = \Delta \cdot \sum_{i=1}^{n-1} \sum_{\substack{j=2 \\ j>i}}^{n} \frac{\eta_{i_M}\eta_{j_M}}{E_{i_M}E_{j_M}}$$

$$D = \Delta \cdot \sum_{i=1}^{n-2} \sum_{\substack{j=2 \\ j>i}}^{n-1} \sum_{\substack{k=3 \\ k>j}}^{n} \frac{\eta_{i_M}\eta_{j_M}\eta_{k_M}}{E_{i_M}E_{j_M}E_{k_M}}$$

$$\vdots$$

$$Y = \sum_{i=1}^{n} E_{i_M} \prod_{\substack{j=1 \\ j\neq i}}^{n} \eta_{j_M}$$

$$Z = \prod_{i=1}^{n} \eta_{i_M}$$

and

$$a = E_{\infty_M} \cdot \Delta$$

$$b = \Delta \cdot \left(E_{\infty_M} \cdot \sum_{i=1}^{n} \frac{\eta_{i_M}}{E_{i_M}} + \sum_{i=1}^{n} \eta_{i_M} \right)$$

$$c = \Delta \cdot \left(E_{\infty_M} \sum_{i=1}^{n-1} \sum_{\substack{j=2 \\ j>i}}^{n} \frac{\eta_{i_M}\eta_{j_M}}{E_{i_M}E_{j_M}} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\eta_{i_M}\eta_{j_M}}{E_{j_M}} \right)$$

$$d = \Delta \cdot \left(E_{\infty_M} \sum_{i=1}^{n-2} \sum_{\substack{j=2 \\ j>i}}^{n-1} \sum_{\substack{k=3 \\ k>j}}^{n} \frac{\eta_{i_M}\eta_{j_M}\eta_{k_M}}{E_{i_M}E_{j_M}E_{k_M}} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\substack{k=1 \\ k\neq i \\ k>j}}^{n} \frac{\eta_{i_M}\eta_{j_M}\eta_{k_M}}{E_{j_M}E_{k_M}} \right)$$

$$\vdots$$

$$y = E_{\infty_M} \sum_{i=1}^{n} E_{i_M} \prod_{\substack{j=1 \\ j\neq i}}^{n} \eta_{j_M} + \sum_{i=1}^{n} E_{i_M}\eta_{i_M} \sum_{\substack{j=1 \\ j\neq i}}^{n} E_{j_M} \prod_{\substack{k=1 \\ k\neq i \\ k\neq j}}^{n} \eta_{k_M}$$

$$z = \left(E_{\infty_M} + \sum_{i=1}^{n} E_{i_M}\right) \prod_{j=1}^{n} \eta_{j_M}$$

A.2 Derivatives of Prony series

In this appendix, the mathematical expressions of the first and second order derivatives of Prony series are shown.

A.2.1 Derivatives as a function of time

A.2.1.1 First order derivatives

Mathematical expressions to calculate maximums or minimums in relaxation modulus and creep compliance as a function of time, using Prony series, are shown in equation A.1 and A.2, respectively.

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -Y_0 \sum_{i=1}^{n} \frac{p_i}{\tau_i} \mathrm{e}^{-\frac{t}{\tau_i}} \tag{A.1}$$

$$\frac{\mathrm{d}J}{\mathrm{d}t} = J_0 \sum_{i=1}^{n} \frac{q_i}{\lambda_i} \mathrm{e}^{-\frac{t}{\lambda_i}} \tag{A.2}$$

A.2.1.2 Second order derivatives

Mathematical expressions to calculate inflexion points in relaxation modulus and creep compliance as a function of time, using Prony series, are shown in equation A.3 and A.4, respectively.

$$\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} = Y_0 \sum_{i=1}^{n} \frac{p_i}{{\tau_i}^2} \mathrm{e}^{-\frac{t}{\tau_i}} \tag{A.3}$$

$$\frac{\mathrm{d}^2J}{\mathrm{d}t^2} = -J_0 \sum_{i=1}^{n} \frac{q_i}{{\lambda_i}^2} \mathrm{e}^{-\frac{t}{\lambda_i}} \tag{A.4}$$

A.2.2 Derivatives as a function of frequency

A.2.2.1 First order derivatives

Mathematical expressions to calculate maximums or minimums in storage modulus, loss modulus, storage compliance, loss compliances and tangents of the phase angle as a function of frequency, using Prony series, are shown in equations A.5-9, respectively.

$$\frac{\mathrm{d}G'}{\mathrm{d}\omega} = 2G_0 \sum_{i=1}^{n} \frac{p_i \tau_i^2 \omega}{(1+\tau_i^2\omega^2)^2} \tag{A.5}$$

$$\frac{\mathrm{d}G''}{\mathrm{d}\omega} = G_0 \sum_{i=1}^{n} \frac{p_i \tau_i (1-\tau_i^2\omega^2)}{(1+\tau_i^2\omega^2)^2} \tag{A.6}$$

$$\frac{\mathrm{d}J'}{\mathrm{d}\omega} = -2J_0 \sum_{i=1}^{n} \frac{q_i \lambda_i^2 \omega}{(1+\lambda_i^2\omega^2)^2} \tag{A.7}$$

$$\frac{\mathrm{d}J''}{\mathrm{d}\omega} = J_0 \sum_{i=1}^{n} \frac{q_i \lambda_i (1-\lambda_i^2\omega^2)}{(1+\lambda_i^2\omega^2)^2} \tag{A.8}$$

$$\frac{\mathrm{d}\tan\delta}{\mathrm{d}\omega} = \frac{\frac{\mathrm{d}G''}{\mathrm{d}\omega}\cdot G' - G''\cdot\frac{\mathrm{d}G'}{\mathrm{d}\omega}}{(G')^2} \tag{A.9}$$

A.2.2.2 Second order derivatives

Mathematical expressions to calculate inflexion points in storage modulus, loss modulus, storage compliance, loss compliances and tangents of the phase angle as a function of frequency, using Prony series, are shown in equations A.10-14, respectively.

$$\frac{\mathrm{d}^2G'}{\mathrm{d}\omega^2} = 2G_0 \sum_{i=1}^{n} p_i \tau_i^2 \frac{1-3\tau_i^2\omega^2}{(1+\tau_i^2\omega^2)^3} \tag{A.10}$$

$$\frac{\mathrm{d}^2G''}{\mathrm{d}\omega^2} = 2G_0 \sum_{i=1}^{n} p_i \tau_i^3 \frac{\omega(\tau_i^2\omega^2-3)}{(1+\tau_i^2\omega^2)^3} \tag{A.11}$$

$$\frac{\mathrm{d}^2J'}{\mathrm{d}\omega^2} = -2J_0 \sum_{i=1}^{n} q_i \lambda_i^2 \frac{1-3\lambda_i^2\omega^2}{(1+\lambda_i^2\omega^2)^3} \tag{A.12}$$

$$\frac{\mathrm{d}^2J''}{\mathrm{d}\omega^2} = 2J_0 \sum_{i=1}^{n} q_i \lambda_i^3 \frac{\omega(\lambda_i^2\omega^2-3)}{(1+\lambda_i^2\omega^2)^3} \tag{A.13}$$

$$\frac{\mathrm{d}^2\tan\delta}{\mathrm{d}\omega^2} = \frac{G'\left(\frac{d^2G''}{d\omega^2}\cdot G' - G''\cdot\frac{d^2G'}{d\omega^2}\right) - 2\frac{dG'}{d\omega}\left(\frac{dG''}{d\omega}\cdot G' - G''\cdot\frac{dG'}{d\omega}\right)}{(G')^3} \tag{A.14}$$

A.3 Roots of polynomial equations

In this appendix, the resolution of cubic and quartic polynomial equations is presented.

A.3.1 Roots of the cubic equation $x^3 + ax^2 + bx + c = 0$

$$x_1 = \frac{1}{12}\left(\frac{-2^{4/3}(1+\sqrt{3}\mathrm{i})(a^2-3b)+2^{2/3}(-1+\sqrt{3}i)H^2-4Ha}{H}\right) \quad (A.15)$$

$$x_2 = \frac{1}{12}\left(\frac{2^{4/3}(-1+\sqrt{3}\mathrm{i})(a^2-3b)-2^{2/3}(1+\sqrt{3}i)H^2-4Ha}{H}\right) \quad (A.16)$$

$$x_3 = \frac{1}{6}\left(\frac{2^{4/3}(a^2-3b)+2^{2/3}H^2-2Ha}{H}\right) \quad (A.17)$$

where

$$H = \sqrt[3]{-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2}} \quad (A.18)$$

A.3.2 Roots of the quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0$

$$x_1 = -\frac{3a+\sqrt{3}\,H_4}{12} - \frac{\sqrt{6}}{12}\cdot\sqrt{3a^2-8b-\frac{2H_3}{\sqrt[3]{H_2}}-\sqrt[3]{4H_2}+\frac{3\sqrt{3}(a^3-4ab+8c)}{H_4}} \quad (A.19)$$

$$x_2 = -\frac{3a+\sqrt{3}\,H_4}{12} + \frac{\sqrt{6}}{12}\cdot\sqrt{3a^2-8b-\frac{2H_3}{\sqrt[3]{H_2}}-\sqrt[3]{4H_2}+\frac{3\sqrt{3}(a^3-4ab+8c)}{H_4}} \quad (A.20)$$

$$x_3 = -\frac{3a-\sqrt{3}\,H_4}{12} - \frac{\sqrt{6}}{12}\cdot\sqrt{3a^2-8b-\frac{2H_3}{\sqrt[3]{H_2}}-\sqrt[3]{4H_2}+\frac{3\sqrt{3}(a^3-4ab+8c)}{H_4}} \quad (A.21)$$

$$x_4 = -\frac{3a-\sqrt{3}\,H_4}{12} + \frac{\sqrt{6}}{12}\cdot\sqrt{3a^2-8b-\frac{2H_3}{\sqrt[3]{H_2}}-\sqrt[3]{4H_2}+\frac{3\sqrt{3}(a^3-4ab+8c)}{H_4}} \quad (A.22)$$

where

$$H_1 = -4(b^2-3ac+12d)^3+(2b^3-9b(ac+8d)+27(c^2+a^2d))^2 \quad (A.3)$$

$$H_2 = 2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{H_1} \quad (A.24)$$

$$H_3 = \sqrt[3]{2}\,(b^2-3ac+12d) \quad (A.25)$$

$$H_4 = \sqrt{3a^2 - 8b + \frac{4H_3}{\sqrt[3]{H_2}} + \sqrt[3]{32H_2}} \tag{A.26}$$

Appendix B

B.1 Relationships with $n = 1$

	GM parameters	relaxation coefficients	creep coefficients	GKV parameters
GM param. known $(E_{\infty_M}, E_{1_M}, \eta_{1_M})$		$Y_0 = E_{\infty_M} + E_{1_M}$ $p_1 = \frac{E_{1_M}}{E_{\infty_M} + E_{1_M}}$ $\tau_1 = \frac{\eta_{1_M}}{E_{1_M}}$	$J_0 = \frac{1}{E_{\infty_M} + E_{1_M}}$ $q_1 = \frac{E_{1_M}}{E_{\infty_M}}$ $\lambda_1 = \eta_{1_M}\left(\frac{1}{E_{\infty_M}} + \frac{1}{E_{1_M}}\right)$	$E_{0_K} = E_{\infty_M} + E_{1_M}$ $E_{1_K} = \frac{E_{\infty_M}}{E_{1_M}}\left(E_{1_M} + E_{\infty_M}\right)$ $\eta_{1_K} = \eta_{1_M}\left(1 + \frac{E_{\infty_M}}{E_{1_M}}\right)^2$
relaxation coeff. known (Y_0, p_1, τ_1)	$E_{\infty_M} = Y_0(1 - p_1)$ $E_{1_M} = Y_0 p_1$ $\eta_{1_M} = Y_0 p_1 \tau_1$		$J_0 = \frac{1}{Y_0}$ $q_1 = \frac{p_1}{1 - p_1}$ $\lambda_1 = \frac{\tau_1}{1 - p_1}$	$E_{1_K} = Y_0 \frac{1 - p_1}{p_1}$ $E_{0_K} = Y_0$ $\eta_{1_K} = Y_0 \frac{\tau_1}{p_1}$
creep coeff. known (J_0, q_1, λ_1)	$E_{\infty_M} = \frac{1}{J_0(1 + q_1)}$ $E_{1_M} = \frac{q_1}{J_0(1 + q_1)}$ $\eta_{1_M} = \frac{\lambda_1 q_1}{J_0(1 + q_1)^2}$	$Y_0 = \frac{1}{J_0}$ $p_1 = \frac{q_1}{1 + q_1}$ $\tau_1 = \frac{\lambda_1}{1 + q_1}$		$E_{0_K} = \frac{1}{J_0}$ $E_{1_K} = \frac{1}{J_0 q_1}$ $\eta_{1_K} = \frac{\lambda_1}{J_0 q_1}$
GKV parameters known $(E_{0_K}, E_{1_K}, \eta_{1_K})$	$E_{\infty_M} = \frac{E_{0_K} E_{1_K}}{E_{0_K} + E_{1_K}}$ $E_{1_M} = \frac{{E_{0_K}}^2}{E_{0_K} + E_{1_K}}$ $\eta_{1_M} = \frac{\eta_{1_K} {E_{0_K}}^2}{\left(E_{0_K} + E_{1_K}\right)^2}$	$Y_0 = E_{0_K}$ $p_1 = \frac{E_{0_K}}{E_{0_K} + E_{1_K}}$ $\tau_1 = \frac{\eta_{1_K}}{E_{0_K} + E_{1_K}}$	$J_0 = \frac{1}{E_{0_K}}$ $q_1 = \frac{E_{0_K}}{E_{1_K}}$ $\lambda_1 = \frac{\eta_{1_K}}{E_{1_K}}$	

B.2 Relationships with $n = 2$

	GM parameters	relaxation coefficients	creep coefficients	GKV parameters
GM parameters known $(E_{\infty_M}, E_{1_M}, E_{2_M}, \eta_{1_M}, \eta_{2_M})$		$Y_0 = E_{\infty_M} + E_{1_M} + E_{2_M}$ $p_1 = \frac{E_{1_M}}{E_{\infty_M}+E_{1_M}+E_{2_M}}$ $p_2 = \frac{E_{2_M}}{E_{\infty_M}+E_{1_M}+E_{2_M}}$ $\tau_1 = \frac{\eta_{1_M}}{E_{1_M}}$ $\tau_2 = \frac{\eta_{2_M}}{E_{2_M}}$	$J_0 = C_{1_M} + C_{2_M}$ $q_1 = \frac{-C_{2_M}(C_{4_M}+C_{5_M}-C_{3_M})}{2C_{4_M}(C_{1_M}+C_{2_M})}$ $q_2 = \frac{-C_{2_M}(C_{4_M}-C_{5_M}+C_{3_M})}{2C_{4_M}(C_{1_M}+C_{2_M})}$ $\lambda_1 = \frac{1}{C_{3_M}-C_{4_M}}$ $\lambda_2 = \frac{1}{C_{3_M}+C_{4_M}}$	$E_{0_K} = \frac{1}{J_0}$ $E_{1_K} = \frac{1}{J_0 q_1}$ $E_{2_K} = \frac{1}{J_0 q_2}$ $\eta_{1_K} = \frac{\lambda_1}{J_0 q_1}$ $\eta_{2_K} = \frac{\lambda_2}{J_0 q_2}$
relaxation coefficients known $(Y_0, p_1, p_2, \tau_1, \tau_2)$	$E_{\infty_M} = Y_0(1 - p_1 - p_2)$ $E_{1_M} = Y_0 p_1$ $\eta_{1_M} = Y_0 p_1 \tau_1$ $E_{2_M} = Y_0 p_2$ $\eta_{2_M} = Y_0 p_2 \tau_2$		$J_0 = C_{1_R} + C_{2_R}$ $q_1 = \frac{-C_{2_R}(C_{4_R}+C_{5_R}-C_{3_R})}{2C_{4_R}(C_{1_R}+C_{2_R})}$ $q_2 = \frac{-C_{2_R}(C_{4_R}-C_{5_R}+C_{3_R})}{2C_{4_R}(C_{1_R}+C_{2_R})}$ $\lambda_1 = \frac{1}{C_{3_R}-C_{4_R}}$ $\lambda_2 = \frac{1}{C_{3_R}+C_{4_R}}$	
creep coefficients known $(J_0, q_1, q_2, \lambda_1, \lambda_2)$		$Y_0 = C_{1_C} + C_{2_C}$ $p_1 = \frac{C_{2_C}(C_{4_C}+C_{5_C}-C_{3_C})}{2C_{4_C}(C_{1_C}+C_{2_C})}$ $p_2 = \frac{C_{2_C}(C_{4_C}-C_{5_C}+C_{3_C})}{2C_{4_C}(C_{1_C}+C_{2_C})}$ $\tau_1 = \frac{1}{C_{3_C}-C_{4_C}}$ $\tau_2 = \frac{1}{C_{3_C}+C_{4_C}}$		
GKV parameters known $(E_{0_K}, E_{1_K}, E_{2_K}, \eta_{1_K}, \eta_{2_K})$		$Y_0 = C_{1_K} + C_{2_K}$ $p_1 = \frac{C_{2_K}(C_{4_K}+C_{5_K}-C_{3_K})}{2C_{4_K}(C_{1_K}+C_{2_K})}$ $p_2 = \frac{C_{2_K}(C_{4_K}-C_{5_K}+C_{3_K})}{2C_{4_K}(C_{1_K}+C_{2_K})}$ $\tau_1 = \frac{1}{C_{3_K}-C_{4_K}}$ $\tau_2 = \frac{1}{C_{3_K}+C_{4_K}}$	$J_0 = \frac{1}{E_{0_K}}$ $q_1 = \frac{E_{0_K}}{E_{1_K}}$ $q_2 = \frac{E_{0_K}}{E_{2_K}}$ $\lambda_1 = \frac{\eta_{1_K}}{E_{1_K}}$ $\lambda_2 = \frac{\eta_{2_K}}{E_{2_K}}$	

Table 2: relationships for models with $n = 2$

With the auxiliary constants:

a) Case "GM parameters known" (1st row in the table) (equations 54 in the text):

$$C_{1_M} = \frac{1}{E_{\infty_M}}$$

$$C_{2_M} = -\frac{E_{1_M}+E_{2_M}}{E_{\infty_M}(E_{\infty_M}+E_{1_M}+E_{2_M})}$$

$$C_{3_M} = \frac{\alpha}{2\eta_{1_M}\eta_{2_M}(E_{\infty_M}+E_{1_M}+E_{2_M})}$$

$$C_{4_M} = \frac{\sqrt{\alpha^2-4E_{\infty_M}E_{1_M}E_{2_M}\eta_{1_M}\eta_{2_M}(E_{\infty_M}+E_{1_M}+E_{2_M})}}{2\eta_{1_M}\eta_{2_M}(E_{\infty_M}+E_{1_M}+E_{2_M})}$$

$$C_{5_M} = \frac{E_{1_M}E_{2_M}(\eta_{1_M}+\eta_{2_M})}{\eta_{1_M}\eta_{2_M}(E_{1_M}+E_{2_M})}$$

where

$$\alpha = E_{\infty_M}\left(E_{1_M}\eta_{2_M} + E_{2_M}\eta_{1_M}\right) + E_{1_M}E_{2_M}\left(\eta_{1_M} + \eta_{2_M}\right)$$

b) Case "Relaxation coefficients known" (2nd row in the table):

$$C_{1_R} = \frac{1}{Y_0(1-p_1-p_2)}$$

$$C_{2_R} = -\frac{p_1+p_2}{Y_0(1-p_1-p_2)}$$

$$C_{3_R} = \frac{\beta}{2\tau_1\tau_2}$$

$$C_{4_R} = \frac{\sqrt{\beta^2-4\tau_1\tau_2(1-p_1-p_2)}}{2\tau_1\tau_2}$$

$$C_{5_R} = \frac{p_1\tau_1+p_2\tau_2}{\tau_1\tau_2(p_1+p_2)}$$

where

$$\beta = \tau_1(1-p_2) + \tau_2(1-p_1)$$

c) Case "Creep coefficients known" (3rd row in the table):

$$C_{1_C} = \frac{1}{J_0(1+q_1+q_2)}$$

$$C_{2_C} = \frac{q_1+q_2}{J_0(1+q_1+q_2)}$$

$$C_{3_C} = \frac{\delta}{2\lambda_1\lambda_2}$$

$$C_{4_C} = \frac{\sqrt{\delta^2 - 4\lambda_1\lambda_2(1+q_1+q_2)}}{2\lambda_1\lambda_2}$$

$$C_{5_C} = \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1\lambda_2(q_1+q_2)}$$

where

$$\delta = \lambda_1(1+q_2) + \lambda_2(1+q_1)$$

d) Case "GKV parameters known" (4th row in the table) (equations 17 in the text):

$$C_{1_K} = \frac{E_{0_K}E_{1_K}E_{2_K}}{E_{0_K}E_{1_K} + E_{0_K}E_{2_K} + E_{1_K}E_{2_K}}$$

$$C_{2_K} = \frac{{E_{0_K}}^2(E_{1_K}+E_{2_K})}{E_{0_K}E_{1_K} + E_{0_K}E_{2_K} + E_{1_K}E_{2_K}}$$

$$C_{3_K} = \frac{\gamma}{2\eta_{1_K}\eta_{2_K}}$$

$$C_{4_K} = \frac{\sqrt{\gamma^2 - 4\eta_{1_K}\eta_{2_K}(E_{0_K}E_{1_K} + E_{0_K}E_{2_K} + E_{1_K}E_{2_K})}}{2\eta_{1_K}\eta_{2_K}}$$

$$C_{5_K} = \frac{{E_{1_K}}^2\eta_{2_K} + {E_{2_K}}^2\eta_{1_K}}{\eta_{1_K}\eta_{2_K}(E_{1_K}+E_{2_K})}$$

where

$$\gamma = E_{0_K}(\eta_{1_K} + \eta_{2_K}) + E_{1_K}\eta_{2_K} + E_{2_K}\eta_{1_K}$$

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