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Inquiry based mathematics education and the development of learning trajectories

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Abstract. This article is based on the panel on inquiry based mathematics education and the development of learning trajectories held at the VARGA 100 Conference. After an introduction presenting the theme and organization of the panel, this article focuses on the diversity of conceptualizations of inquiry based education existing today in mathematics education and their influence on the vision and development of learning trajectories. More precisely, it considers the conceptualizations respectively associated with Realistic Mathematics Education, Genetic Constructivism, Tamás Varga's educational approach and the Anthropological Theory of the Didactic, presented by the panellists, and also shows the efforts undertaken in European projects to reach consensusal visions.

Key words and phrases: Inquiry based mathematics education, Learning trajectory, Realistic mathematics education, Genetic constructivism, Tamás Varga's educational approach, Lajos Pósa method, Anthropological theory of the didactic.

MSC Subject Classification: 97C30Q, 97D10, 97D20, 97D30, 97D40, 97D50.

Introduction

This article is based on the panel on inquiry based mathematics education and the development of learning trajectories held at the VARGA 100 Conference, in Budapest in November 2019, and coordinated by the first author. As was pointed out in the presentation of this panel, in the last decade, a number of projects have been funded to support the large-scale dissemination of Inquiry Based Education in STEM disciplines, in Europe and beyond, and the expressions IBE, Inquiry Based Education and IBL, Inquiry Based Learning, coming from science education, have permeated mathematics education. However, as shown in (Artigue & Blomøj, 2013), in mathematics education, well established didactic traditions that have supported for decades the development of learning trajectories share evident values with the current conceptualizations of IBE and IBL in STEM education. This is also obviously the case for Tamás Varga's approach in terms of complex mathematics education, celebrated and extensively discussed at the conference.

The panel was structured into three phases, each addressing a specific set of questions. In the first phase, the questions at stake were the following ones: How do the conceptualizations of IBME underlying current European projects were established? How do they relate to these traditions? What do they take from these and what do they add to these? Katja Maass, professor at the University of Education in Freiburg and director of the International Centre for STEM Education, recently created in this university, who has to date coordinated nine large-scale European projects in this area, explained how a consensual definition of IBL has been progressively achieved in the project Primas (2010-2013) and how it has been refined in the subsequent projects she has led. Then Michiel Doorman, associate professor at the Freudenthal Institute in Utrecht, who has been part of several of these European projects, situated this consensus with respect to the tradition of "Realistic Mathematics Education", originated in Freudenthal's didactic vision. Ladislav Kvasz, professor at the Department of Mathematics and Mathematics Education of the Pedagogical Faculty at Charles University in Prague, and member of the Institute of Philosophy at the Czech Academy of Sciences, did the same regarding the Czech and Slovakian tradition known as "Genetic constructivism", developed by Vít and Milan Hejný decades ago.

An important question if we want IBME approaches to have a substantial impact on mathematics education is how these approaches support the development of long-term learning progressions, and not only the design of interesting but isolated tasks and situations. This issue was addressed in the second phase of the panel, once again comparing different approaches and their outcomes. First, Péter Juhász, member of the

Alfred Rényi Institute of Mathematics at the Hungarian Academy of Sciences, and teacher at the Szent István High School in Budapest, explained the importance attached to the development of long-term learning progressions in the “Lajos Pósa Method”, an inquiry-based method of teaching directly inspired by Varga’s ideas, used to nurture mathematical talents in informal settings such as week-end camps in Hungary since decades. Then the word was given to Marianna Bosch, professor at the IQS School of Management of the Universitat Ramon Llull, in Barcelona, one of the main contributors to the Anthropological Theory of the Didactic developed by Yves Chevallard. She presented the original vision of IBE that this theory proposes within the paradigm of “Questioning the world”, and the associated conceptualization of learning progressions in terms of “Study and Research Paths”. Both Michiel Doorman and Ladislav Kvasz were then offered to briefly react from their respective perspectives.

The visions and approaches, the examples evoked by the panellists in the first two phases of the panel certainly open fascinating perspectives for mathematics education. However, it is well known that no substantial and sustainable move can be achieved without adequate preparation and support of teachers, and of teacher educators. This is no by chance that the development of resources for the classroom and for teacher professional development, the reflection on adequate strategies for the dissemination of resources and professional development activities, for the development and support of communities of practices, are at the core of most European projects. The last phase of the panel was devoted to this crucial issue, and the reflection was introduced by Katja Maass. She was asked to briefly summarize what she learned from the many projects she has been involved in and from ICSE activities, in that respect. Then time was devoted to interactions with the audience, as had been already the case between the first and second phases.

In this article, we do not reproduce the chronology of the panel but, rather, each panellist presents his/her contribution in a specific section. However, we follow the order of their first presentation in the panel, thus begin with Katja Maass’ contribution, follow with the contributions by Michiel Doorman, Ladislav Kvasz, Péter Juhász and Marianna Bosch, and these five sections are followed by some more global reflections inspired by the panel. These contributions make clear how a diversity of conceptualizations of IBME has emerged from the diversity of theoretical approaches and experiences existing in mathematics education, and the efforts developed to reach definitions consensual enough to make collective work at European level possible.

Defining Inquiry-Based Learning for use in practice (Katja Maass)

There are several definitions of Inquiry-Based Learning (IBL) in the theoretical discussion (NRC 2000; Artigue and Blomhøj, 2013). And among these different interpretations of inquiry-based learning, there are important differences regarding e.g. the degree of autonomy given to students, the objectives pursued with inquiry-based learning (e.g. the development of inquiry competences vs. development of scientific ideas and techniques) and the importance given to real-life questions (Artigue and Blomhøj, 2013).

Whilst in a theoretical discussion it is fruitful to discuss different definitions, we face a different challenge when we intend to develop classroom materials and a common concept for professional development (PD) in an international project. In this case a common definition of IBL is urgently needed. This was the situation in the European project Primas (2010-2013), in which about 30 people from 12 different institutions and 10 countries cooperated. The overall objective of Primas was a widespread implementation of IBL in day-to-day teaching across all partner countries. To this end we developed an international set of tasks and a common concept for a PD course. Considering the objective of the project it was also very important that the definition we found would be understood by teachers. Finding a consensus on a definition was difficult and finally kept us busy for three project meetings (thus, the first project year).

Figure 1. First definition of IBL

One problem was that the definition in the proposal was not very clear. Therefore we decided to define IBL with the help of activities students are supposed to carry out (Fig. 1). This definition appeared to be clear to us because of our hidden understanding of what IBL is and what kind of tasks we mean. However, this definition turned out to be not clear, neither for project partners nor for teachers. Consequently, teachers and school authorities asked for a clear definition as did the advisory panel of Primas. We were also told that other projects (e.g. the Fibonacci project) had similar difficulties.

I would like to illustrate these difficulties with three examples (Fig. 2,3 and 4):

Figure 2. The signing task

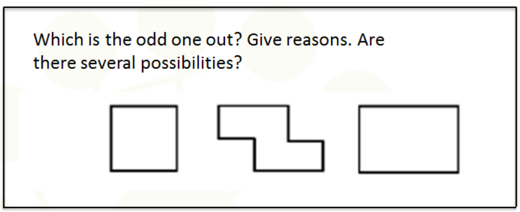

Figure 3. The "Odd one out" task

> Gina wants to buy a new jumper. The original price was 60 Euro. But now it is reduced by 20%.

Figure 4. The jumper task

Whilst some people would say that only the signing task is a task for inquiry, others might argue that the "Odd one out" requires mathematical reasoning and therefore is also an inquiry task. Whilst some might argue that the jumper task is by no means an inquiry task, other would say it is relatively open, because the question is missing. The different perspectives on the tasks might be based in the different cultural background of the different countries and different teaching traditions.

Consequently, we had to find a balance: On the one hand, the definition needed to be a "one fit for all" and should not exclude too many tasks because partners needed to work with the definition in their countries and within their teaching tradition. On the other hand, the definition must allow for selecting a common set of tasks and PD materials. Therefore, within this big project team of about 30 persons we ran a world café and discussed what IBL means to us, what characteristics of inquiry-based learning should be part of the resource and what IBL is not. Based on the world café we came up with a definition outlining different features of IBL (Fig. 5).

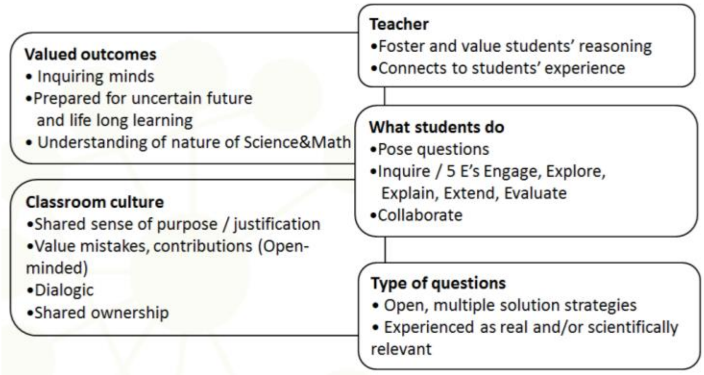

Figure 5. The definition of IBL in Primas

The long time we spend on finding a definition proofed to be worth it, because we used this definition successfully throughout the whole four years of the project. We even continued to use it in later projects, because it was well understood by partners and teachers, our main target group. In our next project (Mascil, 7th Framework programme, 2013-2016), the aim was also a widespread implementation of IBL but this time in connection to the world of work. We used the same definition as shown in Fig. 5, but expanded it to make it even clearer and to include the connection to the world of work (Fig. 6, new aspects typed in italic).

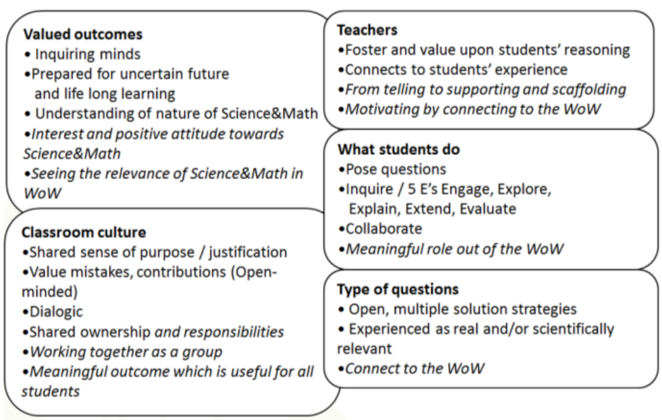

Figure 6. Definition of IBL in Mascil

We kept using the definition of IBL even in the next project (Masdiv, 2017-2020); however, again we expanded the definition. Our approach in Masdiv first introduces IBL as a means for addressing achievement-related diversity to deliver inclusive education and basic skills for all, and thus addresses underachievement. Then, second, it expands IBL to realistic, relevant contexts to promote fundamental values and enhance active citizenship, which also builds social and civic competences and, third, it embeds IBL in multicultural settings to promote intercultural learning.

Summing up, in projects aiming at a widespread implementation of IBL a common definition is important. It is, however, a challenge to find a common definition in large European consortia. It is worth taking some time to find a definition which fits for all.

Relating IBL to Realistic Mathematics Education (Michiel Doorman)

The definition of IBL given above by Katja Maass emphasizes the importance of using types of questions that allow for multiple solution strategies, teachers' trying to connect learning to students' experiences and trying to foster students' inquiring minds. These elements are also connected to and maybe partly rooted in the tradition of Realistic Mathematics Education (RME), originated in Freudenthal's didactic vision on mathematics as an educational task. In addition, recent developments in thinking about how to organize and implement IBL has highlighted how some aspects in RME can be improved or made more explicit. This dialectic relationship is illustrated below with a task that was used in an RME learning trajectory and in the previously mentioned Mascil project.

Freudenthal's vision on mathematics education originates from his critique on an existing practice taking the endpoint of the work of mathematicians as a starting point for instruction (Freudenthal, 1991). The endpoint is a well-organized system of definitions, axioms and theorems that in most cases is the result of a long period of working on problems, relations with other disciplines, generalizing and fine tuning notations and language. When we take that organized system as a starting point for education we kind of invert the route of inquiry. Education starts with definitions and theorems without clear needs for them, and ends with applications that might have been problems from which the mathematical notions originated. This inversion of practice is, according to Freudenthal, anti-didactical, because we don't provide students with any need and purpose for the mathematics we present to them, and maybe even worse, we don't involve them in the work of mathematicians, i.e. in trying to organize phenomena mathematically. As an alternative, he and his colleagues promoted an approach in which learning mathematics is an activity of organizing subject matter from reality (horizontal mathematizing) and generalizing mathematical results of that activity (vertical mathematizing). In this RME approach, learning mathematics starts in problem situations that are meaningful for students such that they experience the need to mathematise them and have opportunities to start reasoning about the situation. By taking mathematics as a human activity for organizing phenomena, the notion of inquiry also becomes a central vehicle for learning.

We illustrate the RME approach and its relation with IBL with a sequence of problems that was part of a learning trajectory used for introducing linear recursive relationships and their converging progression in grade 10. The starting situation concerns some basic information about drug level in a patient's blood while using a certain kind of medicine (Fig. 7).

> A patient is ill. A doctor prescribes a medicine for this patient and advises to take a daily dose of 1500 mg. After taking the dose an average of 25% of the drug leaves the body by secretion during a day. The rest of the drug stays in the blood of the patient.

Figure 7. Initial problem situation

Different questions can be related to this context, for instance: how does the drug level in the blood evolve? When you forget to take a dose one day, is it dangerous to take a double dose the day after? After a collective discussion of the situation during which these questions emerged, students were asked to investigate the progress of the drug level in the blood and to create a leaflet that would provide basic information for the pharmacist who has to inform patients. Students worked in groups during one lesson and had the opportunity to present the next lesson their results and the information they wanted to include in this leaflet. The open task was expected to elicit multiple solution strategies that would let them experience the structure of recurrence relationships, use words to talk about such relationships, and create the need for more systematic procedures and notations for describing their calculations. The work of the students showed a diversity in approaches and also in the final converging level of the drug in the blood (Fig. 8).

Some students wrote systematically all the calculations, while others quickly organized them in tables, or even used graphs or formulas. This variety in approaches provided an opportunity to formulate general characteristics of the approaches and of the underlying calculation procedures. Moreover, some of the groups also felt the need to describe the progress of the drug level with a ‘traditional’ functional relationship. In the middle of the work of the left group of Figure 8 can be seen how they tried to formulate such a functional relationship between days x and the increase of the drug level y. Next, the rather big differences observed between the converging levels obtained created the need to question the calculation procedures, in this case either next = 0,75·current + 1500 resulting in a converging level of 6000 mg, or next = 0,75·(current +1500) resulting in a converging level of 4500 mg. During these activities and subsequent lessons language and notations developed, as well as final definitions of recurrence relationships and their convergence.

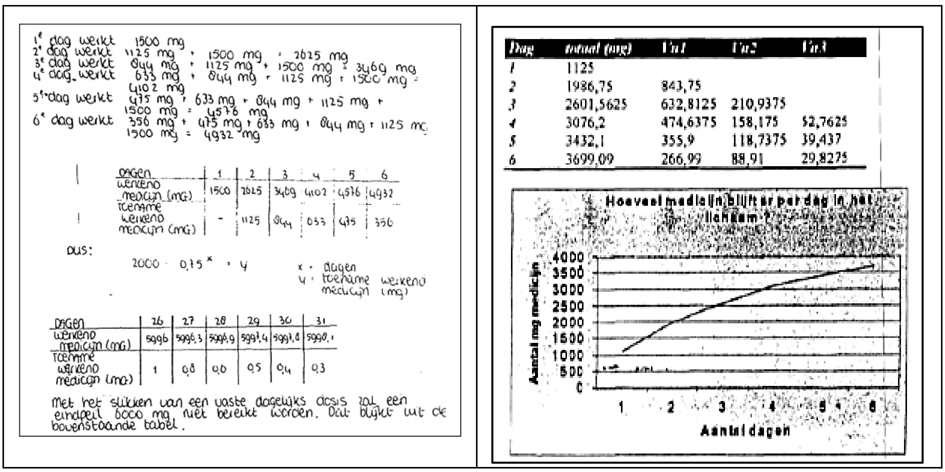

Figure 8. Student work of two groups on the Drug level task

The above example illustrates the RME approach, but also some shared visions between RME and IBL. Both approaches are student centred and foster the use of open non-routine problem situations to offer opportunities for mathematising and for processes of inquiry (e.g. questioning, experimenting systematically and communicating results). However, as processes of inquiry were not paid explicit attention in RME, the opportunities for reflecting on these processes might not have been taken. During EU-projects like Primas and Mascil, we further developed the potential of the Drug level task to illustrate how such tasks can be a resource for addressing IBL explicitly (Doorman, Jonker and Wijers, 2016). One option is to discuss with teachers the difference between the above open version of the task and a very structured textbook version (Fig. 9). The difference is obvious as the structured version provides all questions, all information needed and a stepwise solution procedure. The context can be neglected, students do not see alternative approaches and the need of each step, and the teacher receives limited information about what students can do when they do not have the provided steps at hand.

- How much mg of the drug is in the blood of the patient after one day?
- Finish the table.

Day	mg of drug in the blood
0	0
1	1125
2	
3	

- Explain why you can calculate the amount of drug for the next day with the formula: new_amount = (old_amount + 1500) · 0,75
- After how many days has the patient more than 4 g medicine in the blood? And after how many days 5 g?
- What is the maximum of amount of the drug that can be reached?

Figure 9. Structuring the Drug level task

However, the unstructured version of the task also has the risk that students feel lost and get frustrated, or that parts of the task will ask so much time from the students that they are unable to reach a reasonable result within the timeframe of the lesson. To prevent this to happen, the teacher has a role in structuring the lesson. Consequently, the unstructured version of the task needs a carefully structured lesson plan to scaffold the students' inquiry. Another option to involve students in the inquiry process - with a more structured task - is to cut the structured version into pieces and present all sub-questions in a different order, or as pieces of a jigsaw and ask the students to find the original textbook order. After deciding upon a sensible order, they would have the chance to reflect on the stepwise structure of their textbook and get the opportunity to reflect on the role of systematizing calculation procedures by using tables and formulas. These alternative versions of one task appeared to be a valuable resource for discussing IBL with mathematics teachers, for creating opportunities to let them experience its potential and to extend their teaching repertoire.

In retrospect, RME helped to understand how open and non-routine problems for IBL can also be used for content learning in topic related learning trajectories. These trajectories have attention for horizontal and vertical mathematization processes changing focus from situation-specific solutions to conventional concepts and procedures. IBL helped us to communicate with teachers why and how to explicitly address mathematization, problem solving and generalization processes. Although mathematization is at the heart of RME, a reflection on these processes has the risk of being neglected in daily teaching. Moreover, the presence of a definition of inquiry-based learning in the EU projects helped teachers to create attention for more general processes of inquiry, like questioning, planning, modelling and experimenting systematically. Students of all ages have curious minds and can inquire. We need to use and foster that potential in mathematics education.

The six principles of Genetic constructivism (Ladislav Kvasz)

Genetic constructivism was developed by Vít and Milan Hejný, father and son. Vít Hejný (1904 – 1977) was a secondary school teacher of mathematics. He started to develop a new method of mathematics education in total isolation in Stalinist Czechoslovakia in the fifties. He lived in a small city, Martin, in central Slovakia. His son Milan (*1936), who was trained as a mathematician, joined his father in educational efforts in the early seventies, when horrified by the way his son was taught mathematics, he decided to teach mathematics in the class his son was attending. The contact with mathematics teaching engrossed him to such a degree that he abandoned his mathematics career and became involved in teacher training.

In 1985 Tamás Varga visited Milan Hejný in Bratislava and taught in his primary school class. As Milan Hejný remembered, for about twenty minutes the children needed translation of what Varga was saying (the interpreter was one of the children speaking Hungarian), but gradually a kind of mutual understanding without words developed. Afterwards Varga invited Milan Hejný to teach in his primary school class in Budapest. The premature death of Tamás Varga in 1987 prevented this collaboration to bring more tangible fruits than a paper (Hejný 2007).

After the split of Czechoslovakia in 1992 Milan Hejný moved to Prague and started to implement the method developed by his father into a series of primary school textbooks. In Czech Republic the Ministry of education only checks whether a textbook fulfils the requirements concerning the basic goals, but schools are free to choose any of the approved textbooks. Currently about 20% of the children are taught mathematics according to the textbooks written by the team led by Milan Hejný.

Vít Hejný was not a mathematician but a teacher; the method he developed has some features in common with other approaches developed by mathematicians, but some of them are different (see Bachratý et al., 2012, 2016). I will present the ***cognitive*** principles of *Genetic constructivism*. In general this method is an *Inquiry Based Learning* method in the sense that students are discovering mathematics by their own activity. The method consists mostly in posing questions, it fosters student´s reasoning, encourages collaboration, appreciates mistakes as a basis for further inquiry and discussion and it develops a dialogic classroom culture. These characteristics are in line with those outlined above by Katja Maass. Nevertheless, it is specific by taking the social as well as historical dimension of mathematics seriously. The problems follow the general process of development of mathematics.

1. The principle of epistemic closeness of mathematics. Thinkers like Plato, Descartes, Kant, or Brouwer claimed that mathematical knowledge has a special status – it is inborn, *a priori* or intuitive. It is not important to decide who is correct. The very existence of these views indicates that mathematical knowledge is **epistemologically close**. If someone wants to find out where is the spring of Nile, he must travel to Africa. In this sense geographical knowledge is epistemologically distant. Mathematical knowledge is accessible to immediate experience and authentic mathematical knowledge emerges from student’s own experience acquired in the process of his/her own cognitive activities in contact with mathematical reality.

This principle is shared with many reformers inspired by constructivism. The role of the teacher is not to tell the students how things are but to pose problems, so that they are supposed to solve these problems autonomously. In *Genetic constructivism* the textbooks contain almost only problems, which the children are supposed to solve. So the teacher totally draws himself back.

2. Principle of ontological grounding of mathematics. There are thinkers like Aristotle, Galileo, or Arnold, who stress the connection of mathematics with reality. Arnold famously said that mathematics is a part of physics where experiments are cheap. Again, our task is not to decide who is right. The very existence of such views indicates that mathematics is ***ontologically grounded***. If we consider language, which is *epistemologically close* just like mathematics, i.e. a native speaker has immediate access to the entire richness of linguistic meaning, the difference is that language is grounded in social conventions. That 2 + 2 = 4 is not an outcome of any such social conventions but a fact rooted in the very way how things in the world are. Of course, the language in which we formulate mathematical results has a conventional component. But the fact that for instance the group of rotations in three dimensional space is noncommutative is a fact that is independent from the conventions of the language we use when formulating and proving it.

In *Genetic constructivism* this principle has deep consequences. The teacher is not allowed to say the children if they make a mistake, he is only allowed to show them a related problem, where they can find the mistake themselves. The point is that if the teacher takes on the role of judge who decides what is right and wrong, the children start to develop strategies to please the teacher. The answer of a question "why is xxx yyy" of the form "because the teacher said so" is wrong, because it replaces the *ontological grounding* of mathematics by its *social grounding*. So the role of the teacher is shifted to a rather difficult role. He has to know for any mistake the children make, problems or questions that make it possible for the children to discover their mistake themselves.

3. The principle of instrumental anchoring of mathematics. Even if the basic facts of arithmetic are immediately accessible, and it is possible to count without a positional system, as the Egyptian or Roman numerals show, if one wants to find the product of two 7-digit numbers, it is much better to have a positional system at one's disposal. Mathematics is interested not only in the basic truths, but also in complex situations. For dealing with them mathematicians developed ***representational instruments*** such as the decimal positional system, the notational system of polynomial algebra, or fractions; these allow solving complicated problems. Mathematics educators agree that one task of mathematics education is to teach children to use the standard representational tools of mathematics.

Genetic constructivism is radically different. In genetic constructivism we do not teach representational tools. Instead we develop several so called ***environments***. For instance, for addition these include the number line drawn on the floor of the classroom, where the children can walk and find out, how many is 7 (steps) plus 8 (steps). This environment has the advantage that minus 7 means to walk seven steps backwards. And – (–7) means turn back and walk 7 steps backwards. So in this context it is clear why minus times minus is plus. Another environment for addition is the bus. There are several stations; at each station some passengers get off and some get on and the children have to find out how many passengers are on the bus at the terminal station. There are other environments for addition, and they all are isomorphic (even if not totally, because – (–7) has no meaning in the context of the bus). When children discover the isomorphism of the environments the right moment has come to show them the representational tool for addition. The same happens with algebra, there are again several more or less isomorphic environments in the framework of which similar problems are solved. Only when the children realize that they need not to solve a problem in the next environment, because the solution of the algebraically isomorphic must be the same as in the previous one, they are prepared to learn the algebraic notation. Thus the conventional rules are introduced only after the children understand the mathematics behind them. This is so, because the main *danger* that is lurking behind the work with the representational tools is that children take the conventional rules to be of the same importance as the mathematical principles behind them.

4. The principle of discursive anchoring of mathematics. The main feature that distinguishes mathematics from other discipline is the idea of a *proof*. Thus to teach mathematics means to teach proving. In *Genetic constructivism* this means that the children have to learn to find, formulate, evaluate and defend arguments in discursive intercourse among each other. The teacher must not tell, which argument is correct and which not. That must be decided by the children themselves. If the entire class accepts a wrong argument, that only means that the class is not ready. The task of the teacher is not to question, criticize or to correct the decision of the class but to wait and after some time come with a similar problem, in which the mistake in the argument would come to the fore. Further the teacher has to make sure that every argument gets heard and that the argumentation happens in a friendly cooperative atmosphere.

5. The genetic principle. The genetic principle means that the cognitive development of the child happens in phases and that the problems formulated by the teacher have to take into account the state reached by the children. Genetic constructivism is based on the belief that the cognitive development of the child copies the main features of the historical development of mathematics. For more detailed exposition and discussion of the genetic principle see (Schubring, 1978).

Genetic constructivism is like an iceberg; on the surface we see only a series of problems. We do not see the enormous amount of research engaged into the selection and formulation of these problems. This work means first to study carefully the historical development of the particular area of mathematics and to identify the basic developmental stages. Then the series of mathematical problems is tested with children, in order to find suitable series of problems, the solution of which can lead children with different mathematical abilities from one stage to the next, up to the final level. The next task is to identify all epistemological obstacles, all misconceptions and other difficulties, that resurface during the testing. And finally additional problems are searched for, problems that enable the child to successfully surmount these difficulties.

6. The historical principle. Mathematics is an integral part of human culture and of human history. Teaching of mathematics by reinventing the main conceptual shifts in the historical development of mathematics has the potential to reintegrate the children into a culture of rational discourse and acceptance of the views of others. The children learn that truth is the outcome of discussion, dialogue and argumentation. It further fosters in the children the understanding of the cultural heritage of western science and enhances the creative potential of self-development.

7. Further non-cognitive principles. Besides these *cognitive* principles Genetic constructivism has also several educational goals. I can list only some of them. 1. To give the weaker students the experience of enjoying intellectual work by means of including problems of various levels of difficulty. 2. To reduce the fear of making mistakes and to show that a mistake can be the beginning of a new learning process by not penalizing the mistakes. 3. To teach to have joy from the success of others by fostering team work. 4. To systematically teach to recognize the difference between demagogy and rational reasoning by letting the children discuss the solutions they find. 5. To let the students experience a culture of friendly discussion and fairness in the class.

Conclusion. *Genetic constructivism* can be seen as an IBL approach that bases the learning trajectories of children on a thorough reconstruction of the historical development of mathematics. The research is used mostly for understanding the cognitive background of mathematical knowledge.

Discovery learning: the Pósa method (Péter Juhász)

Historical background

Lajos Pósa is an outstanding teacher especially engaged in the nurturing of Hungarian mathematical talents (Győri & Juhász, 2018). Born in 1947, he was a child prodigy and silver and gold medalist in International Mathematical Olympiad in 1965 and 1966. He co-authored 4 papers with Pál Erdős, the first one when he was 15. After getting a PhD in graph theory Pósa turned to teaching mathematics. In the beginning of the 1970s, Pósa joined the “complex mathematical education” movement in Hungary led by Tamás Varga (Gosztonyi, 2018). In the 1980s, he became a member of the Mathematical Didactics Research Group at the Alfréd Rényi Institute of Mathematics, led by János Surányi, studying the applicability of the Varga method in high schools. Between 1982 and 1991, he taught two high school classes for 4 years each, during which he developed teaching materials with the aim of making the learning of mathematics enjoyable, focusing not on the mechanical application of formulas and algorithms, but on relaxed and cheerful, autonomous and logical thinking.

He then turned to talent care, which has a well-established tradition in Hungary, with a national network of special mathematics circles for students, the organization of several ‘high-quality’ competitions, and a mathematics journal for students, the KöMaL published since 1893 (Frank, 2012). However, Pósa felt that the school environment, with maximum 90 minutes for study circle sessions did not enable students to be involved in the intense thinking that highly talented students need. Besides he did not consider preparing for competitions to be at the core of nurturing talent, but rather that students think on exciting, interesting problems and discover the beauties of mathematics autonomously.

Pósa launched a talent nurturing weekend math camp for one single group in 1988. Camp organization has been evolving continuously since then; for instance, in the early times the main activity was students’ individual autonomous work, together in a large room; now they work in groups of 2-4 students in separate rooms. While in the first camps Lajos Pósa was the only teacher and organizer, there are presently 2-7 ‘assistant teachers’ offering help to the ‘camp leader’. One group has 2 or 3 camps a year, and follows the program during 6 years (Juhász, 2019).

The main goal of teachers implementing the Pósa method is to give students the opportunity to dive into the beauty of mathematics using and improving their creativity, problem solving skills, and perseverance. Providing enough time for the students to find a solution, a good definition or an exciting follow-up question is an essential feature of the method. The teacher supports the construction of a new “building” in the students’ mind; the bricks are the problems, new definitions, and challenging follow-up questions, the walls are the different strategies and approaches to solve problems. The “construction” spreads over 4-5 years and many weekend camps with many homework problems given to students between them.

Thus the Pósa method shares evident characteristics and values with IBL as defined above: the central role given to the formulation and solving of significant problems and questions, the autonomy given to students in the inquiry process, the importance attached to the quality of their mathematical activity. However, beyond the fact that it has been developed for nurturing mathematical talents in informal settings, the attention paid to the aesthetic dimension of mathematics and the organization of problems along long term threads make it specific. We focus below on the special characteristics of problem threads.

Problem threads in the Pósa method

A problem thread in the Pósa method refers to a series of tasks the teacher proposes to students. Students have the real opportunity to solve interesting and challenging problems. The tasks are carefully sequenced and selected; they build on each other like building blocks, gradually guiding students to learn a particular mathematical idea. Furthermore, these tasks serve as vehicles for fostering students' reasoning, problem solving, and communication skills. Tasks can be of many different types: a mathematical problem to be solved, posing a question or developing a definition (which helps students think like mathematicians), playing a game to find a winning strategy or to discover some kind of connection. Threads typically span over long periods of time, often several years. Many tasks simultaneously belong to different threads, thus creating a complex network.

We focus here on one example: the thread called "Proof of Impossibility". One of the most exciting parts of mathematics, indeed, is that we can prove that something is impossible. We cannot construct something, or we cannot arrange numbers with some given property, etc. A priori, there are a lot of options to try, but with mathematical tools it can be proved that it is impossible. This breath-taking property has the potential to impress even young students. To give the reader a taste, we selected four problems from this thread.

Problem 1. Draw at least three from the following four maps: there are exactly a) 6, b) 7, c) 8, d) 9 towns on the map and exactly 3 highways go out from every town (each highway connects two towns without passing through others).

Students often ask questions while they are working, for example: is it allowed to connect two towns with more than one highway? Is it required that we can reach every town from every other town?, etc. It turns out that it is not so difficult to draw maps with 6 and 8 towns. However, students struggle with the other two tasks. And there is a good reason! These maps are impossible to create. We always stress to our students the interest and importance of being able to prove this impossibility. This shows the strength of mathematics.

Problem 2. One corner of a standard chessboard is removed. Is it possible to cover the remaining 63 squares with 21 triominos? (A triomino is a 3×1 rectangle)

Before we propose this problem, the students have solved the well-known problem where two opposite corners of a chessboard are removed and the question is: is it possible to cover the remaining 62 squares with 31 dominos? To solve this problem it is enough to use the traditional colouring of the chessboard. To solve the triomino problem, students need to improve the solution of the previous problem, because the traditional colouring does not help. If the students understand why the solution of the previous problem worked, then they can find how to colour the board with 3 colours to prove that the required covering does not exist (see Fig. 10). Believing that 3 colours can help to solve the problem is the first step, and after this idea they need to find a proper colouring. (Note that there are other ways to prove the impossibility in both problems.)

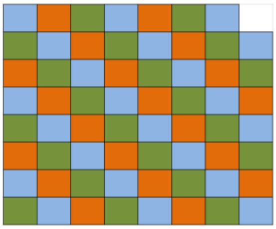

Figure 10. A possible chess colouring

The thread Proof of Impossibility contains even more interesting problems involving boards or other figures which can be solved with appropriate adaptations of this colouring strategy.

Problem 3. Given ten integers on the circumference of a circle, we can replace two neighbouring numbers with their average. Is it always possible to make all ten numbers equal using this operation?

Lot of students think that mathematics is an abstract science with no room for experimentation. But this is absolutely not true. Although experimentations cannot prove a general statement, they can make easier to find a conjecture and understand the whole situation. In the Pósa method we emphasize that making experiments in mathematics is very much welcome. We have another long thread; the title is Experimentation, Conjecture, Proof. This third problem contributes to this thread, too.

Trying some steps and calculating the average of adjacent numbers shows that if we start with integers, we only get rational numbers whose denominators are powers of 2 since if you take two numbers with this property and calculate their average, the denominator of the result will be a power of 2. Considering this and the fact that the sum of the numbers does not change during the process, we can easily find an initial situation from where it is not possible to reach the desired situation.

Another important goal of the method is to teach students how to pose good questions. After solving this problem, the next task is to collect interesting questions. A very natural and easy way to pose good questions is trying to generalize the problem. We can ask now what we can tell if we have n numbers on the circumference of the circle. Using the above mentioned idea it is not so difficult to show that if n is not a power of 2, we can always find a counterexample. It is obvious that if there are only 2 numbers then we can make them equal. The next step is more or less obvious, starting with 4 integers we can always make them equal. So the first really interesting question is: what happens if we have 8 numbers?

This is one of the most difficult problems in our math camps.

Problem 4. Given a line (e) and a point P not on (e) in the plane, is it possible to construct a line perpendicular to e, such that P lies on it, using only a straightedge?

Before we pose this question students have solved a sequence of problems where we want to understand how many times we need to use a compass to construct a line which is perpendicular to a given line. Then we let them discover the rich area of geometrical transformations in the plane, and at last we (or they) pose the question: does there exist a geometrical transformation in the plane which preserves lines but not angles. If they have solved this problem and know that such a transformation exists, they can work on the construction problem. We do not mention them that the existence of such a transformation can help. After a while we give them this information as a hint if necessary. Teachers should be very careful with this problem because it is very hard to figure out the solution, despite the fact that understanding the proof is not so difficult.

The thread “Proof of Impossibility” is a very extensive, colourful selection of tasks from very different traditional topics of mathematics. We think that these tasks serve the original purpose because they are interesting problems, students need to make real efforts to solve them, and during the time they work on them they can improve their creativity, problem solving strategies and perseverance.

Study and research paths in the paradigm of questioning the world (Marianna Bosch)

The Anthropological Theory of the Didactic (ATD) was developed by Yves Chevallard from the eighties, as an extension of his theory of didactic transposition (see (Bosch & Gascón, 2005) for an historical presentation). More recently (Chevallard, 2015, 2019), this researcher proposed to approach teaching and learning processes by considering two main paradigms. The first one is the *paradigm of visiting works*, which currently prevails in many educational systems. In it, the curriculum is proposed as a collection of contents – or *works* – teachers know in advance and organise for the students to learn them. Learning processes can be seen as guided visits to curricular contents where the teacher presents or introduces the topics to the students who have to know them and be able to do something with them. Some of the curriculum works are treated as real "monuments", historical creations important enough to declare its visit as compulsory by all citizens. The "monumentalisation" of curriculum contents is part of today's educational crisis.

To avoid assuming the paradigm of visiting works as the only possible one, we consider a broader paradigm that includes the visit of works but radically modifies its role. We call it the *paradigm of questioning the world*. In this paradigm, the curriculum is not defined as a collection of works to study, learn or "visit", but as a set of questions to address, study and answer. This paradigm is only partially present at school, and in a very frugal and mitigated way. It is however dominant in other social institutions, like academic research for instance, but not only. We can consider police work, journalistic inquiries and lawyers' investigations are also part of it.

A key difference between both paradigms is the role played by questions and answers. In the visiting works, students learn answers, that is, knowledge organisations that have been historically elaborated to answer questions that might have lost their meaning today, or that we have just forgotten. These answers come first and, sometimes, teachers try to find problems to motivate their use. In this perspective, questions are subordinated to the works – the answers. In the paradigm of questioning the world, questions go first: one has to consider them as significant by themselves and approach them to elaborate an answer by any means, not by using this or that previously established work.

In the ATD, we use the two paradigms as scientific models, that is, as tools to analyse the educational reality by pointing at some of its specificities. We are especially interested in the conditions that maintain the old paradigm in vigour and prevent the new one to develop. Our research team has been working during more than ten years in the implementation and analysis of teaching and learning processes that try to include as many elements as possible of the paradigm of questioning the world (Bosch 2018). The instructional format used is what we call *study and research paths* (SRPs).

Let us introduce the main elements of an SRP by considering an example taken from (Barquero, Monreal, Ruiz-Munzón & Serrano, 2018). The starting point of an SRP is an open question Q_0 a community of study – made by students X and teachers Y – decide or are required to address. In the example, the *generating question* Q_0 was proposed in 2016 to a group of first-year university students in a degree of economics and management: "A research developed by Princeton University in 2014 predicted that Facebook would lose 80% of its users before 2017. Can this forecast be true? What kind of forecast would you propose about Facebook users for the short and long term?" To answer Q_0, the study community raises some *derived questions* Q', Q'', Q''', ... they find easier to address: about the Princeton study and the methods used, the data directly available, the definition of Facebook users, the possible forecast strategies they can perform, etc. The partial answers obtained generate new derived questions in what is called a *questions-answers* dialectic, which creates a self-sustained dynamic in the SRP (Bosch & Winsløw, 2015) that ends up with the elaboration of a final answer $A^{\heartsuit}$ to Q_0 (Fig. 11).

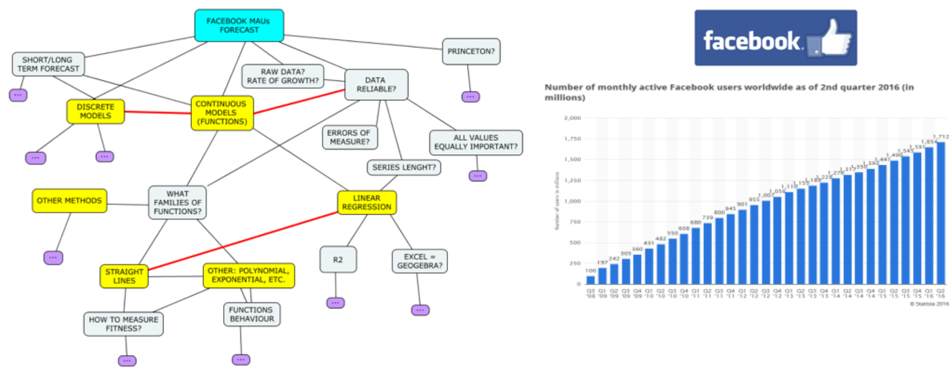

Figure 11. Example of questions-answers map and of data available about Facebook users (Barquero et al., 2018)

The students can answer some of the derived questions by using the means available in their *milieu*: previous knowledge about Facebook, Excel sheet to manage the collected data, elementary functions to model them, etc. Other questions will need some new information that can be searched in the *media* that comprises all the accessible information sources. In this case, the media included the Internet, but also books, papers, other teachers, etc. The search for information leads the students to find some works other people have created and organised to provide answers to other questions, some of them similar to Q_0, like the Princeton study; other only partially related, like the forecast techniques, the data presentations, etc. (Fig. 11). These works must be *searched* and *studied*, that is, deconstructed and reconstructed to provide specific answers to the derived questions. The milieu will then be enriched through what we call the *media-milieu dialectic*. For instance, the collected data will be modelled by different functional relationships (linear, exponential, etc.), raising the question about which of the proposed models is better and in what sense. Then the forecasts can be compared to the Princeton study, leading to the question of the assumptions made by each strategy. The process finishes when the community of study considers that its answer $A^\heartsuit$ to Q_0 can be presented, defended and disseminated according to the initial requirements and time constraints.

The inquiry process followed during an SRP is usually represented with what we call the *Herbartian schema* $[S(X; Y; Q_0) \rightarrow M] \hookrightarrow A^\heartsuit$ (Chevallard, 2019). In this schema, $S(X; Y; Q_0)$ designates the study community made of the students X and the teachers Y around question Q_0. The process aims to provide the final answer $A^\heartsuit$ and, for this, the study community has to build an appropriate milieu M. This milieu includes the previously available tools and knowledge and all the other elements generated and found during the inquiry: derived questions, empirical data, external works (or answers to other questions) produced by others' inquiries, etc. One can use this schema to describe the elements and dynamics of SRPs – or any inquiry process –, to question them and point at their limitations.

Many questions can be raised thanks to the Herbartian schema, about the elements integrated into the milieu and their evolution, about the role of teachers and students in the managing of these elements, and especially about the elements that are not in the milieu, about possible study and research gestures students and teachers could do and do not, etc. For instance, it can be shown that in many inquiry-based instructional strategies, the media-milieu dialectic is only partially developed. In particular, the searching of new information and already available answers, their study and integration into the milieu, or their rejection if the study does not reveal anything important about them, these kinds of gestures are rarely put to the fore. And teachers and students do not always have the appropriate resources to do it. The management of the questions-answers dialectic also reveals important difficulties: students are not used to formulating derived questions, deciding which one to address first, planning the whole process, searching in the media, questioning the information found and validating it, etc. And teachers do not have many resources to help them with it. For instance, the study community often lacks a terminology to designate the derived questions, the sources of information, the provisional answers, the dead ends, the useless information, etc., because, in the paradigm of visiting works, technical terms are usually reserved to the official and important works – the "monuments".

The design and implementation of SRPs are critical to analysing two main phenomena. On the one hand, we find many constraints that hinder the development of SRPs, and that we attribute to the prevalence of the paradigm of visiting works. For instance, the difficulty of taking Q_0 seriously; the teachers' temptation to show what they think is the best path because it leads to use more important pieces of knowledge, etc. On the other hand, we study the conditions and instructional devices that favour the transition to the new paradigm. In the case of the questions-answers dialectic, we have seen that the use of questions-answers maps (Fig. 11) appears to be a useful tool for teachers and students to plan and manage the inquiry process: it provides ostensive materiality to the lack of terminology previously mentioned. A lot of progress is still necessary, and many problems remain open, especially those related to the choice of the generating questions and, consequently, to the kind of final answers considered as acceptable, to their evaluation and destiny. If an inquiry process has to take Q_0 seriously, then, as a consequence, the final answer $A^{\heartsuit}$ has to become something important and useful, not just the end of the inquiry. What to do with these final answers within and outside the community of study, how to assess them and by whom, how to make them available for further use, all these open questions are difficult to answer, given today's school prevailing pedagogy and epistemology.

Nevertheless, the investigations about SRPs carried out also show that many aspects of the related inquiry processes can be carried out, at least partially, under the school and university conditions. To mention only one, it is sometimes remarkable to see the unexpectable fruitfulness of many generating questions, when one lets the students follow their own paths and take their research seriously. Searching in the media usually provides pieces of information to decipher that generate much more interesting questions that those initially forecasted by the designers. The kind of works – and monuments – that need to be mobilised to address them also go beyond the controlled universe of knowledge that is organised at school. Finding the way to integrate this kind of inquiry processes in today's educational systems remains a big challenge that still needs a lot of effort and research.

Reflection and conclusion

As shown by the different sections of this text, different didactic traditions have developed in Europe, based on the vision of mathematics as the product of activities that human beings develop to answer questions arising from the world around them, and to better understand it, and with the conviction that this epistemological vision should also be at the basis of the teaching of this discipline and the mathematics that pupils and students experience in schools. Another important point is that these different traditions pay specific attention to the fact that mathematics is a highly connected field as stressed by Felix Klein in his famous lectures to teachers long time ago, thus also the importance of thinking not in terms of isolated sets of questions and answers, but in terms of long term teaching and learning trajectories. However, these contributions also show that such foundational principles have nurtured and still nurture a diversity of theoretical constructions and practical realizations, according to contexts and cultures, and to the particular profile and experience of those at their origin. For instance, despite the central role given to inquiry processes in all approaches evoked in this text, the visions of learning they rely on, the role given to the teacher in the learning process, are quite different. Genetic constructivism combines a radical constructivist vision of learning, *a priori* giving to the teacher a very limited role, and a genetic view seeing in the historical development of mathematical concepts a fundamental source for organizing learning trajectories. In ATD, the inquiry process is modelled in term of *Herbartian schema*, and according to this schema teacher and students constitute a community in charge of finding jointly an answer to the question at stake. Moreover, as made clear by the description and example, the selection of questions is not driven by genetic concerns. Differences are also clear when considering the logic underlying the design of long term learning and teaching trajectories in RME, in the Pósa method and in ATD, as made clear by the provided examples. In RME, the starting points are most often realistic extra-mathematical situations and learning trajectories are thought in terms of processes of horizontal and vertical mathematizations; in the Pósa method, the threads of problems developed over several years obey another logic, and as shown by the example, the mathematization of extra-mathematical situations does not play a major role. In ATD, trajectories depend on the succession of questions that will emerge from the process of study and inquiry on an initial question, and are quite open.

As made clear by Katja Maass in her contribution, developing the level of consensus necessary for making international collaboration possible and for capitalizing on the knowledge produced through different approaches, is not an easy task. However, understanding better our similarities and differences, how these impact our didactic strategies and realizations, with what consequences, understanding how we can mutually enrich from the existing diversity, is something important, and we hope that this text will contribute to the efforts already made in this direction. One of its originality is that it combines the contributions of traditions that, for historical reasons, remained rather separated for several decades. Genetic constructivism or the Hungarian tradition of mathematics education are not so much familiar to many researchers from Western Europe involved in research on IBME, and conversely many researchers from Eastern Europe are not so familiar with RME, and above all ATD. In fact, the efforts made here can be related to those undertaken since 15 years now and captured by the expression of "networking of theories" (Bikner-Ashbahs & Prediger, 2014) for limiting the risk of fragmentation of the field of mathematics education, due to the exponential increase of theoretical constructions, and for improving communication and capitalization of knowledge. These offer conceptual tools and methodologies which could be helpful for systematizing the work initiated in the panel at the Varga 100 Conference reported in this text.

References

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM – The International Journal on Mathematics Education, 45(6)*, 797- 810.

Bachratý, H et al. eds. (2012 and 2016). *Archív Víta Hejného* (Archive of Vít Hejný), vol. I. and II. EDIS, Žilina.

Barquero, B., Monreal, N., Ruiz-Munzón, N., & Serrano, L. (2018). Linking transmission with inquiry at university level through study and research paths: The case of forecasting Facebook user growth. *International Journal of Research in Undergraduate Mathematics Education, 4(1)*, 8-22.

Bikner-Ahsbahs, A., & Prediger, S. (Eds.) (2014). *Networking of theories as a research practice*. Switzerland: Springer.

Bosch, M. (2018). Study and Research Paths: a model for inquiry. In B. Sirakov, P. N. de Souza, & M. Viana (Eds.), *International Congress of Mathematicians* (Vol. 3, pp. 4001–4022). Rio de Janeiro: World Scientific Publishing Co. Pte. Ltd

Bosch, M., Winsløw C. (2015). Linking problem solving and learning contents: the challenge of self-sustained study and research processes. *Recherches en Didactique des Mathématiques*, *35 (2)*, 357-401.

Chevallard, Y. (2015). Teaching Mathematics in tomorrow's society: a case for an oncoming counter paradigm. In S. J. Cho (Ed.), *The proceedings of the 12th international congress on mathematical education* (pp. 173-187). Springer, Cham.

Chevallard, Y. (2019). On using the ATD: Some clarifications and comments. *Educação Matemática Pesquisa, 21(4),* 1-17

Császár, Á. (2005). Education and Research in Mathematics. In J. Horváth, (Ed.), *A Panorama of Hungarian Mathematics in the Twentieth Century. Bolyai Society Mathematical Studies 14. Budapest Berlin [etc.], János Bolyai Mathematical Society* (pp. 555–562). Springer.

Doorman, L.M., Jonker, V., & Wijers, M.M. (2016). Mathematics and Science in Life: Inquiry Learning and the World of Work. University of Education Freiburg. https://www.mascil-project.eu/images/pdf/Mascil_BOOK_EN_web.pdf

Frank, T. (2012). Teaching and learning science in Hungary, 1867–1945: Schools, personalities, influences. *Science and Education, 21(3)*, 355–380.

Freudenthal, H. (1991). Revisiting mathematics education. Dordrecht: Kluwer.

Gosztonyi, K., Kosztolányi J., Pintér, K., Vancsó, Ö. & Varga, E. (2018). Varga's «complex mathematics education» reform: at the crossroad of the New Math and Hungarian traditions. In Y. Shimizu & R. Vithal (Eds.), *ICMI Study 24: School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities*. Conference proceedings (pp 133-140). https://drive.google.com/file/d/1za-Jlip112xg53NrZ1szjAOK3rOeTWc0/view

Győri, J. G., & Juhász, P. (2018). An extra-curricular gifted support programme in Hungary for exceptional students in mathematics. In K. S. Taber, M. Sumida, & L. McClure (Eds.), *Teaching gifted learners in STEM subjects: Developing talent in science, technology, engineering and mathematics* (pp. 89-106). New York, NY: Routledge.

Hejný, M. (2007). Graphs as an environment. *A 2004 – 2006. Évi Varga Tamás napok előadásai*. Nyitott Könyvműhely, Budapest, 43-54.

Juhász, P. (2019). Talent Nurturing in Hungary: The Pósa Weekend Camps. *Notices of the American Mathematical Society*, *66(6)*, 898-900.

Maaβ, K., & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: a synthesis. *ZDM – The International Journal on Mathematics Education, 45(6)*, 779- 795.

National Research Council (2000). Inquiry and the National Science Education Standards. A guide for teaching and learning. Washington, DC: National Academy Press.

Schubring, G. (1978). *Das genetische Prinzip in der Mathematik-Didaktik*. Klett, Stuttgar.

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